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Critical percolation in the plane: conformal invariance, Cardy's formula, scaling limits

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Abstract. In this Note we study critical site percolation on triangular lattice. We introduce harmonic conformal invariants as scaling limits of certain probabilities and calculate their values. As a corollary we obtain conformal invariance of the crossing probabilities (conjecture attributed to Aizenman by Langlands, Pouliot, and Saint-Aubin in [7]) and find their values (predicted by Cardy in [4], we discuss simpler representation found by Carleson). Then we discuss existence, uniqueness, and conformal invariance of the continuum scaling limit. The detailed proofs appear in [10]. © 2001 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Percolation critique dans le plan : invariance conforme, formule de Cardy, objets limites

Résumé. *Dans cette Note, nous nous intéressons à la percolation critique par sites sur le réseau plan triangulaire. Nous introduisons des invariants conformes harmoniques et nous montrons qu'ils correspondent à la limite, lorsque la maille du réseau tend vers zéro, de probabilités d'événements discrets. En particulier, nous obtenons l'invariance conforme asymptotique des probabilités de croisement et la formule de Cardy. Dans un second temps, nous étudions l'existence, l'unicité et l'invariance conforme d'objets limites.* © 2001 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

Nous étudions la percolation critique par sites sur le réseau planaire triangulaire. En d'autres termes, chaque sommet du graphe est colorié (indépendamment des autres) avec une couleur parmi deux possibilités, par exemple bleu et jaune, avec probabilité $1/2$. On s'intéresse alors aux propriétés macroscopiques des composantes connexes du sous-graphe formé par les sites d'une couleur fixée (par exemple bleue). Comme référence générale, citons le livre [5] (*voir* aussi [1–4,7] et les références dans [10]).

Invariants conformes harmoniques. – La propriété-clé des probabilités associées à la percolation que nous considérons ici est le fait qu'elles dépendent de manière harmonique d'un paramètre z variant dans un

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domaine plan. Cette propriété permet de les déterminer de manière unique à partir de leur comportement au bord du domaine, et implique qu'ils sont invariants par transformations conformes.

On peut prédire l'existence de tels invariants conformes harmoniques en supposant l'existence et l'invariance conforme d'une limite asymptotique (lorsque la maille du réseau tend vers zéro) de la percolation, voir [10]. Nous considérons ici un invariant particulier : la complexification des probabilités de croisement. Soit Ω un triangle topologique avec sommets (ou bouts) désignés par $a(1), a(\tau), a(\tau^2)$ dans le sens inverse des aiguilles d'une montre ($\tau := \exp(2\pi i/3)$). Pour un réseau triangulaire de maille δ, nous définissons les fonctions $H_\alpha^\delta(z)$ (constantes sur les triangles du réseau), $\alpha \in \{1, \tau, \tau^2\}$, $z \in \Omega$, comme les probabilités de l'événement qu'il existe un chemin simple (c'est-à-dire auto-évitant) bleu, joignant l'arc $a(\alpha)a(\tau\alpha)$ à l'arc $a(\tau^2\alpha)a(\alpha)$ et séparant z de l'arc $a(\tau\alpha)a(\tau^2\alpha)$.

Pour un triangle equilatéral Ω' avec sommets $a'(1), a'(\tau), a'(\tau^2)$, nous définissons les fonctions linéaires h'_α, $\alpha \in \{1, \tau, \tau^2\}$, valant 1 en $a'(\alpha)$ et 0 aux deux autres sommets. Pour un triangle topologique quelconque Ω nous définissons les fonctions h_α comme les composées des h'_α par l'application conforme $\Omega \to \Omega'$, $a(\alpha) \mapsto a'(\alpha)$.

THÉORÈME 1. – *Lorsque $\delta \to 0$, les fonctions H_α^δ convergent uniformément dans Ω vers les fonctions h_α. En particulier, leurs limites sont des invariants conformes des points $a(1), a(\tau), a(\tau^2), z$ et du domaine Ω.*

Si le point z est choisi sur l'arc de bord $a(\tau^2)a(1)$, alors $H_{\tau^2}^\delta(z)$ donne la *probabilité de croisement* – la probabilité d'avoir une composante connexe bleue qui connecte les arcs $za(1)$ et $a(\tau)a(\tau^2)$.

COROLLAIRE 1 (formule de Cardy, version de Carleson). – *Lorsque $\delta \to 0$, la probabilité de croisement est invariante conforme. Pour un triangle equilatéral de côtés de longueur un, elle tend vers $|za(1)|$.*

Pour démontrer le théorème 1, nous montrons que les fonctions H_α^δ sont asymptotiquement harmoniques, et qu'elles sont solution du même problème de Dirichlet–Neumann que les h_α. L'harmonicité est établie en montrant que les H_α^δ forment un « triplet harmonique conjugué ». Il se trouve que leurs dérivées discrètes sont données par les probabilités de certains événements « à 3 branches ». Les événements correspondant à $\frac{\partial}{\partial\eta}H_\alpha$ et à $\frac{\partial}{\partial\tau\eta}H_{\tau\alpha}$ sont différents, mais ils ont la même probabilité, ce que implique les équations de Cauchy–Riemann (2).

Les objets limites. – Il serait logique d'anticiper l'existence d'un objet continu, sorte de configuration limite de la percolation sur le réseau, lorsque la maille tend vers zéro. Cet objet serait alors invariant conforme et vérifierait la formule de Cardy. Les résultats connus de compacité [2] impliquent l'existence de limites faibles pour des sous-suites. Notre théorème implique qu'une telle limite faible vérifie la formule de Cardy, ce qui semble la déterminer de manière unique et la forcer à être invariante conforme. Une description complète de la configuration limite semble être inévitablement technique (*cf.* [1]) ; la preuve sera contenue dans la suite de [10].

Une description possible de la percolation dans le plan se fait via une collection des courbes fermées emboîtées, les périmètres des composantes connexes des deux couleurs. Dans [10] nous démontrons que la loi d'un périmètre (normalisé) tend vers une limite. Soit Ω un triangle topologique avec trois sommets a, b et c désignés dans le sens inverse des aiguilles d'une montre. Pour chaque configuration de la percolation, il existe une unique courbe γ^p, le « périmètre », qui suit les arêtes du réseau dual hexagonal et relie le sommet a à l'arc bc, en séparant les composantes connexes bleues touchant ab des composantes connexes jaunes touchant ca.

THÉORÈME 2. – *Lorsque $\delta \to 0$, la loi μ_δ^p du périmètre discret converge faiblement vers une loi, μ^p, concentrée sur les chemins Hölder, avec des points doubles* (*mais sans intersection propre*), *de a à bc. Cette loi est un invariant conforme de la configuration (Ω, a, b, c).*

Dans la preuve de théorème 2 nous n'utilisons que des propriétés de la loi μ^p (l'invariance conforme, la propriété locale et la formule de Cardy, *cf.* [11]) qui sont valables pour le processus de Schramm SLE_6 cordale [9], ou pour l'enveloppe du mouvement brownien réfléchi (Werner [11]).

COROLLAIRE 2. – *La loi μ^p coïncide avec la loi du SLE_6 cordal. La loi de l'enveloppe de γ^p coïncide avec la loi de l'enveloppe du mouvement brownien réfléchi.*

Notons que les valeurs de plusieurs dimensions fractales pour SLE_6 sont connues (*voir* Lawler, Schramm et Werner, [8]). Les résultats analogues pour la percolation en découlent immédiatement.

We discuss critical site percolation on triangular lattice: vertices are independently colored in two colors, say blue and yellow, with equal probability $1/2$, and properties of clusters (maximal connected subgraphs of fixed color) are investigated. For general background on percolation consult the book [5], for topics related to this paper see [1–4,7] and other references in [10].

Harmonic conformal invariants. – We start by considering percolation-related quantities whose key property is their harmonic dependence on a point $z \in \Omega$. It allows us to determine them uniquely from their boundary behavior, and forces them to be conformally invariant. Harmonic conformal invariants related to Brownian motion were introduced by Kakutani in [6]. There are several harmonic conformal invariants related to percolation, and one can predict their existence assuming existence and conformal invariance of percolation scaling limit, see [10]. We consider a particular invariant which is a complexification of crossing probabilities.

Take a topological triangle – a simply connected domain Ω with three accessible boundary points (or prime ends), labeled counterclockwise $a(1), a(\tau), a(\tau^2)$ $(\tau := \exp(2\pi i/3))$. For a triangular lattice with mesh δ, we define (constant on the lattice triangles) function $H_\alpha = H_\alpha^\delta(z)$ to be the probability of an event $Q_\alpha(z),\ \alpha \in \{1, \tau, \tau^2\},\ z \in \Omega$, which is an occurrence of a blue simple path connecting the arcs $a(\alpha)a(\tau\alpha)$ and $a(\tau^2\alpha)a(\alpha)$, and separating z from the arc $a(\tau\alpha)a(\tau^2\alpha)$.

For an equilateral triangle Ω' with vertices $a'(1), a'(\tau), a'(\tau^2)$, we define linear functions $h'_\alpha,\ \alpha \in \{1, \tau, \tau^2\}$, to be 1 at the vertex $a'(\alpha)$ and 0 at remaining vertices. For a general topological triangle Ω with vertices $a(1), a(\tau), a(\tau^2)$, we define $h_\alpha,\ \alpha \in \{1, \tau, \tau^2\}$, to be the pull-backs of linear functions h'_α under the conformal map $\Omega \to \Omega'$, $a(\alpha) \mapsto a'(\alpha)$.

THEOREM 1. – *As $\delta \to 0$, functions H_α^δ converge uniformly in Ω to functions h_α. Particularly, their limits are conformal invariants of the points $a(1), a(\tau), a(\tau^2), z$ and the domain Ω.*

We prove the Theorem by showing that H_α^δ's are harmonic in the limit and satisfy the same mixed Dirichlet–Neumann problem as h_α's. The harmonicity is established by finding a harmonic conjugate and checking that contour integrals vanish (this is easier than working with Laplacian, which seems hardly possible). Interestingly, instead of a pair of harmonic conjugate functions, we get a "harmonic conjugate triple" h_1, h_τ, h_{τ^2}. It seems that $2\pi/3$ rotational symmetry enters in our paper not because of the specific lattice we consider, but rather manifests some symmetry laws characteristic to (continuum) percolation.

When point z is chosen on the boundary arc $a(\tau^2)a(1)$, then $H_{\tau^2}^\delta(z)$ gives the *crossing probability*, i.e. the probability of having a blue cluster connecting the arc $za(1)$ to the arc $a(\tau)a(\tau^2)$.

COROLLARY 1 (Cardy's formula in Carleson's form). – *In the limit as $\delta \to 0$, the crossing probability is conformally invariant. For an equilateral triangle with side length one it tends to $|za(1)|$.*

Continuum scaling limit. – It is logical to anticipate that there is a continuum object, corresponding to scaling limit (as mesh tends to zero) of the lattice percolation, which is conformally invariant and satisfies Cardy's formula. One can represent a discrete percolation configuration as a collection of "nested" oriented closed curves – perimeters of clusters of both colors. These curves are the unique curves along the edges of the dual lattice, separating clusters of opposite colors, and in the limit they will be the only curves corresponding to crossings by both colors. In the discrete case such curves will be simple, in the scaling limit they cease to be simple but remain "self-avoiding" (i.e. without "transversal self intersections"). Alternatively one can work with Aizenman's proposition (*see* [1]) to represent a given percolation configuration by a collection of all curves inside all clusters of some fixed color.

Known compactness results (*see* [2] by Aizenman and Burchard) imply existence of weak subsequential limits of the laws of collections of all perimeters (as measures on the space of Hölder curves collections). Our result implies that any such weak limit satisfies Cardy's assertions, which seems to determine it uniquely and force conformal invariance. The rigorous proof is likely to be quite technical (one can attempt, e.g., to retrieve collection of all perimeters by induction, using Theorem 2) and will be the subject of a follow up paper.

In this paper we will sketch the proof that one (properly normalized) perimeter has a scaling limit. Consider some domain Ω with three points (or prime ends) a, b, and c on the boundary, named counterclockwise. For any percolation configuration there is a unique perimeter curve γ^p along the edges of the dual hexagonal lattice, which goes from a to bc separating the blue clusters intersecting the arc ab from the yellow clusters intersecting the arc ca.

THEOREM 2. – *As $\delta \to 0$, the law μ^p_δ of the discrete perimeter converges weakly to a law μ^p on Hölder self-avoiding* (*but non-simple*) *paths from a to bc. This law is a conformal invariant of the configuration* (Ω, a, b, c).

Alternatively one can define γ^p in the continuum setting as restriction of a full percolation configuration to a coarser σ-algebra. Note also, that in the scaling limit the "hull" of the curve γ^p is formed by the boundaries (inside Ω) of all yellow percolation clusters intersecting the arc ac, viewed from b, and all blue percolation clusters intersecting the arc ab, viewed from c. Boundary (also called "external perimeter") is understood in topological sense, when cluster is regarded as a compact subset of the plane. Perimeter differs from the boundary, since it enters "zero width" fjords.

In the proof that the subsequential limit of the laws μ^p_δ is uniquely determined, we only use its properties (conformal invariance, locality, and Cardy's formula, *cf.* [11]) valid for the Schramm's chordal SLE_6 process, started at a and aiming at bc (*see* [9]). Similarly, the hull of γ^p satisfies all the properties characterizing the hull of the reflected Brownian motion (started at a, reflected on ab and ac at $\frac{\pi}{3}$-angle pointing towards bc, and stopped upon hitting bc, see Werner [11]).

COROLLARY 2. – *The law μ^p coincides with that of the Schramm's chordal SLE_6. The law of the hull of γ^p coincides with the law of the hull of reflected Brownian motion* (*or chordal SLE_6*).

The values of various dimensions and exponents for SLE_6 are known by the work of Lawler, Schramm, and Werner (*see* [8]), and hence similar results for percolation follow immediately.

Proof of Theorem 1. – Take $\beta \in \{1, \tau, \tau^2\}$. Let z be the center of some lattice triangle, η be a vector from z to the center of one of the adjacent triangles. We denote by $P_\beta(z,\eta)$ probability of the event $Q_\beta(z+\eta) \setminus Q_\beta(z)$. Then the discrete derivative of H_β can be written in terms of P_β's:

$$\frac{\partial}{\partial\eta} H_\beta(z) := H_\beta(z+\eta) - H_\beta(z) = P_\beta(z,\eta) - P_\beta(z+\eta, -\eta). \tag{1}$$

LEMMA 1 ($2\pi/3$-Cauchy–Riemann equations). – *One has* $P_\beta(z,\eta) = P_{\tau\beta}(z, \tau\eta)$. (2)

One can deduce from (2) that the discrete Cauchy–Riemann equations for H_α's hold up to δ^ε.

To prove Lemma 1, name the vertices of the triangle which contains z by the letters X, Y, Z starting with the one opposite to $z+\eta$ and going counterclockwise. If the event $Q_\beta(z+\eta) \setminus Q_\beta(z) =: Q$ occurs, there should be a blue simple path γ going from the arc $a(\beta)a(\tau\beta)$ to the arc $a(\tau^2\beta)a(\beta)$ and separating z from $z+\eta$. So there are two disjoint blue paths ("halves" of γ), which go from Y and Z to the arcs

$a(\tau^2\beta)a(\beta)$ and $a(\beta)a(\tau\beta)$ correspondingly. Also the vertex X is is joined by a simple yellow path to the arc $a(\tau\beta)a(\tau^2\beta)$, otherwise $Q_\beta(z)$ occurs.

But in the latter description we can change the colors of the paths preserving the probability of Q, *cf.* [3]. In fact, condition on the area Ω' bounded by the "counterclockwise-most" yellow path from X to the arc $a(\tau\beta)a(\tau^2\beta)$, the "clockwise-most" blue path from Y to the arc $a(\tau^2\beta)a(\beta)$, and containing $a(\tau^2\beta)$. Then the probability of the existence of a blue path from Z to $a(\beta)a(\tau\beta)$ is the same as the probability of such a yellow path (since inverting colors in $\Omega \setminus \Omega'$ preserves the probability of a configuration). Taking expectation over all possible Ω''s, we deduce, that the event Q has the same probability as an occurrence of three disjoint simple paths, joining X, Y, and Z to the arcs $a(\tau\beta)a(\tau^2\beta)$, $a(\beta)a(\tau^2\beta)$, and $a(\beta)a(\tau\beta)$, and colored yellow, blue, and yellow correspondingly. This is the description of the event $Q_{\tau\beta}(z+\tau\eta)\setminus Q_{\tau\beta}(z)$, and we proved Lemma 1.

LEMMA 2 (Hölder norm estimates). – *There are constants ε and C depending on the domain Ω only, such that H_β has ε-Hölder norm at most C. The boundary values of H_β are zero on the arc $a(\tau\beta)a(\tau^2\beta)$ and $H_\beta(a(\beta)) \to 1$ as $\delta \to 0$. Also $H_1 + H_\tau + H_{\tau^2}|_{\partial\Omega} \to 1$ as $\delta \to 0$.*

Lemma 2 is an easy consequence of the Russo–Seymour–Welsh theory (*cf.* [5], 11.70).

Now take some equilateral triangular contour Γ with bottom side parallel to the real axis and vertices $x(1), x(\tau), x(\tau^2)$ (named counterclockwise, starting with the top one) at the centers of some lattice triangles. For a function $H(z)$ we define the discrete contour integral $\oint_\Gamma^\delta H(z)dz$ as an appropriate Riemann sum over centers of lattice triangles lying on Γ.

LEMMA 3 (Contour integrals vanish). – *For any equilateral triangular contour $\Gamma \subset \Omega$ of length ℓ with bottom side parallel to the real axis and any $\beta \in \{1, \tau, \tau^2\}$ one has*

$$\oint_\Gamma^\delta H_\beta^\delta(z)\,dz = \oint_\Gamma^\delta \frac{1}{\tau} H_{\tau\beta}^\delta(z)\,dz + \mathrm{O}\big(\ell\,\delta^\varepsilon\big).$$

The proof is similar to the usual proof that contour integrals of analytic functions vanish: we sum (2) over the interior of the contour (instead of integrating the usual Cauchy–Riemann equations), using "telescoping sums" and (1).

By Lemma 2 the Hölder norms of functions H_α^δ, $\alpha \in \{1, \tau, \tau^2\}$, are uniformly bounded, hence from any sequence of such functions with $\delta \to 0$ one can chose a uniformly converging subsequence. Therefore to show that the functions H_α^δ converge uniformly, as $\delta \to 0$, to the functions h_α and prove Theorem 1, it is sufficient prove the following

LEMMA 4. – *Assume that for some subsequence $\delta_j \to 0$ the functions $H_\alpha^{\delta_j}$ converge uniformly in Ω to some functions f_α. Then $f_\alpha \equiv h_\alpha$.*

To prove Lemma 4 observe that the discrete contour integrals $\oint^{\delta_j} H_\beta^{\delta_j}$ converge to the usual contour integrals $\oint f_\beta$. Then by Lemma 3 for any equilateral triangular contour $\Gamma \subset \Omega$ of length ℓ, with bottom side parallel to the real axis, and any $\beta \in \{1, \tau, \tau^2\}$ one has

$$\oint_\Gamma f_\beta(z)\,dz = \oint_\Gamma \frac{1}{\tau} f_{\tau\beta}(z)\,dz. \tag{3}$$

Summing equation (3) for $\beta = \alpha$ and equation (3) for $\beta = \tau\alpha$, multiplied by $-\big(\frac{1}{2} + \frac{i}{2\sqrt{3}}\big)$, and applying Morera's theorem, we obtain that f_α has harmonic conjugate $\frac{1}{\sqrt{3}}(f_{\tau\alpha}(z) - f_{\tau^2\alpha}(z))$.

It follows that the functions f_α are harmonic and satisfy for any unit vector η

$$\frac{\partial}{\partial\eta} f_\alpha = \frac{\partial}{\partial(\tau\eta)} f_{\tau\alpha}. \tag{4}$$

By Lemma 2 the boundary values of f_α are equal to zero on the arc $a(\tau\alpha)a(\tau^2\alpha)$ and to 1 at the point $a(\alpha)$. If ν is the counterclockwise-pointing unit tangent to $\partial\Omega$ then (4) implies that $\frac{\partial}{\partial(\tau\nu)} f_\alpha = \frac{\partial}{\partial(\nu)} f_{\tau^2\alpha} = 0$ on the arc $a(\alpha)a(\tau\alpha)$ and similarly $\frac{\partial}{\partial(-\tau^2\nu)} f_\alpha = 0$ on the arc $a(\tau^2\alpha)a(\alpha)$. The same assertions are clearly valid for h_α's, thus f_α's and h_α's satisfy the same mixed Dirichlet–Neumann problem, which has a unique solution and we have proven Lemma 4 and Theorem 1.

Proof of Theorem 2. – Theorem A.1 in [2] implies that the family of laws $\{\mu_\delta^p\}_\delta$ of perimeters on mesh δ lattices is weakly precompact in the space of Hölder curves with the uniform metric, so we can find a weak-$*$ subsequential limit μ^p. It is sufficient to show that the latter law is independent of the chosen subsequence.

First we note that the law of the hull of γ^p is uniquely determined. Indeed, probability of the event $A_{\eta,\eta'}$ of the hull being contained between two disjoint simple curves η (going from ab to bc) and η' (going from bc to ca) is uniquely determined by the Cardy formula. On the other hand, such events generate (by disjoint unions and complements) the Borel σ-algebra of hulls.

To pass to the law μ^p, for a fixed ϵ one defines a broken line approximation $\gamma_\epsilon = \phi_\epsilon(\gamma)$ to some a parameterization γ of a perimeter curve γ^p by induction: $\gamma_\epsilon(t_{j+1}) := \gamma(t_{j+1})$, where t_{j+1} is the time of the first exit of $\gamma[t_j, 1]$ from the ball $B(\gamma(t_j), \epsilon)$. The law of the hull (viewed from b) of $\gamma[0, t_{j+1}]$ coincides with the law of the union of the hull of $\gamma[0, t_j]$ with the hull of γ^p inside $B(\gamma_j, \epsilon) \setminus \mathrm{hull}(\gamma[0, t_j])$. The latter is uniquely determined, hence by induction the laws of the hulls and so of the endpoints of the broken line are uniquely determined as well.

Thus the law $\phi_\epsilon^{-1}(\mu^p)$ is uniquely determined. Observing that γ_ϵ converges to γ (with appropriate estimates), we conclude that w-$\lim_\epsilon \phi_\epsilon^{-1}(\mu^p)$ is equal to μ^p, which is therefore uniquely determined. Conformal invariance is automatic.

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