Interrelations between some ergodic theorems

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The ergodic theorem

- Let (Ω, μ) be a measure space with $\mu(\Omega) = 1$, and $L : \Omega \to \Omega$ a measure preserving map, i.e. $\mu(L^{-1}A) = \mu(A)$ for all measureable $A \subset \Omega$. Suppose L is μ-ergodic and let f be an integrable function. It was an important question to know whether

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- In the special case of independence of the terms, this is known as the weak/strong law of large numbers, in probability theory:

$$\frac{1}{n}(X_1 + X_2 + ... + X_n) \to E(X_i),$$

first version of which was proved by J. Bernoulli towards the end of the 17th century. Modern version due to the works of Markov, Chebyshev, Borel, Cantelli, Khintchine, Kolmogorov.

Part I:
On von Neumann's Ergodic Theorem

The von Neumann ergodic theorem

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- The theorem holds true for any linear operator U of a Hilbert space, assuming the norm $\|U\| \leq 1$ and there are simple proofs.

Another reformulation

- One can take one step further and define for any $g \in \mathcal{H}$, $\phi(g) := Ug + f$. Then ϕ is an isometry (or merely semicontraction in the case $\|U\| \leq 1$).

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$$\phi^3(0) = U(U(U0 + f) + f) + f = U^2 f + Uf + f$$
etc. Hence
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etc. Hence
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- Pazy proved in 1971 that for any semicontraction $\phi : \mathcal{H} \to \mathcal{H}$, i.e. map $\mathcal{H} \to \mathcal{H}$ such that $\|\phi(x) - \phi(y)\| \leq \|x - y\|$, it holds that there is a vector $v \in \mathcal{H}$ such that
$$\frac{1}{n}\phi^n(0) \to v$$
in norm. (v is the vector in the closure of all displacements $\phi(x) - x$ closest to 0).

Yet another reformulation

- There is a unit speed geodesic $\gamma(t) = tv/\|v\|$ in $\mathcal{H}$ such that

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- Suppose $Z_n(\omega)$ is an integrable ergodic cocycle of semicontractions of a complete metric space X which is Buseman nonpositively curved and uniformly convex (e.g. CAT(0) or uniformly convex Banach space). Then K.-Margulis proved in 1999 that there is a number $l \geq 0$ and unit speed geodesic ray γ (depending measurably on ω) such that

$$\frac{1}{n}d(Z_n(\omega)x_0, \gamma_\omega(nl)) \to 0.$$

A corollary: a random mean ergodic theorem

- Let $f \in L^2(Z, \lambda)$. Pitt, von Neumann-Ulam, Kakutani considered

$$\frac{1}{n}\sum_{k=0}^{n-1} f\left(T_{L^k\omega} T_{L^{k-1}\omega} \ldots T_{\omega} z\right)$$

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- This can again be reformulated into unitary operators or into a cocycle of isometries:

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- That "the middle expression" converges was already known: *the random mean ergodic theorem of Beck-Schwartz* from 1957.

Another special case: isometries of CAT(0) spaces

Corollary

Let ϕ be an isometry of a complete CAT(0)-space and with positive displacement $\inf_{x \in X} d(\phi(x), x) > 0$. Then for any x the sequence $\phi^n(x)$ converges to a point $\xi \in \partial X$ as $n \to \infty$ (and which must be fixed by ϕ).

Proof.

- Let $a_n = d(\phi^n x, x)$. One has $a_{n+m} \leq a_n + a_m$.

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- For any sequence $\varepsilon_i \to 0$ one can choose n_i such that

$$a_{n_i} - a_{n_i - k} \geq (l - \varepsilon_i) k$$

for all $1 \leq k < n_i$.

Proof.

- Consider the the triangles x, $\phi^{n_i}x$ and $\phi^{n_j}x$. With ε_i small enough the geodesic segments from x form a "Cauchy sequence". From which we get a limit geodesic ray with the desired property. It is unique.

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- Recall that this in particular gives a proof of the original von Neumann's theorem.

Linear functionals

- Kohlberg & Neyman 1981: Let X be a Banach space and $\phi : X \rightarrow X$ a semicontraction. Then there exists a linear functional f of norm 1 such that
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- I proved a few years ago that this theorem also holds more generally for a cocycle $Z_n(\omega)$ of semicontractions of a *reflexive* Banach spaces.
- Conjecture: reflexiveness can be removed. (This would have implications for ergodic cocycles taking values in C*-algebras)
- Question?: Let ϕ be a semicontraction of a Hilbert space. Then is there a linear functional f such that $f(\phi^n(0)) \geq 0$ for all $n > 0$? (In finite dimensions this was proved by Beardon).

Part II : On Birkhoff's Ergodic Theorem

Birkhoff's ergodic theorem

- Let (Ω, μ) be a measure space with $\mu(\Omega) = 1$, and $L : \Omega \to \Omega$ a measure preserving map, i.e. $\mu(L^{-1}A) = \mu(A)$ for all measureable $A \subset \Omega$. Suppose L is μ-ergodic.

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$$\frac{1}{n}\sum_{k=0}^{n-1} f(L^k\omega) \to \int_\Omega f(\omega)d\mu(\omega) \text{ for } \mu\text{-a.e. } \omega.$$

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- This appeared in Proc N.A.S. in December 1931 while von Neumann's paper appeared in Proc N.A.S. in January1932.

Recall a few theorems extending Birkhoff

Oseledec multiplicative ergodic theorem, mid 1960s $GL(n, \mathbb{R})$; Kaimanovich, mid 1980s, symmetric spaces of nonpositive curvature, Gromov hyperbolic spaces; K.-Margulis 1999.

Theorem (K.-Ledrappier, 2006)

Let $Z_n(\omega)$ be an integrable ergodic cocycle taking values in a group G which acts by isometry on a proper metric space X. Then for a.e. ω there is a horofunction $h_\omega \in \partial X$ such that

$$\lim_{n\to\infty} -\frac{1}{n} h_\omega(Z_n(\omega)x_0) = l = \lim_{n\to\infty} \frac{1}{n} d(Z_n(\omega)x_0, x_0).$$

- Birkhoff is the case $G = \mathbb{R}$ acting by translation on $(X, d) = (\mathbb{R}, |\cdot|)$, $x_0 = 0$. The horofunctions $\partial(\mathbb{R}, |\cdot|) = \{h_{+\infty}(x) = -x, h_{-\infty}(x) = x\}$. Moreover, $Z_n(\omega) = f(\omega) + f(L\omega) + ... + f(L^{n-1}\omega)$ and $l = \left|\int_\Omega f(\omega) d\mu\right|$.

$$\lim \pm\frac{1}{n}\left(f(\omega) + f(L\omega) + ... + f(L^{n-1}\omega)\right) = \left|\int f(\omega) d\mu\right|$$

Reflexive metric spaces

- Buseman: Let γ be a geodesic ray from x_0 in a metric space X. Then

$$h_\gamma(y) = \lim_{n\to\infty} d(\gamma(n), y) - d(\gamma(n), x_0)$$

is a point in the geometric boundary ∂X (defined by Gromov around 1980), which was the new points of the closure of the image of X under

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- When does every $h \in \partial X$ arise in this way?
- Definition? A complete metric space is reflexive if

$$\partial_{rays} X \cong \partial X.$$

Ok if X is UC & US. E.g. CAT(0)-spaces.

Limit sets

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- Another application of the K-Margulis/K-Ledrappier theorem, in view of Kaimanovich's ray criterion for the identification of Poisson boundaries:
- **Theorem:** Let Γ be a finitely generated nonamenable group acting properly on a CAT(0)-space with bounded geometry. Then the limit set must be uncountable.

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- "$\Longleftarrow$" is not true in the ergodic setting. "$\Longrightarrow$" Sawyer in the 1960s.
- **Corollary (Aaronson)** Let L be a positive function such that $L(0) = 0$, $L(t) \to \infty$, L is subadditive and $L(t)/t \to 0$. If f is a measurable function $\mathbb{R} \to \mathbb{R}$ such that

$$\int_\Omega L(|f|)d\mu < \infty, \text{ then } \frac{1}{n} L\left(\left|\sum f(L^k\omega)\right|\right) \to 0 \text{ as } n \to \infty.$$

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- **Our proof:** The space $(\mathbb{R}, L(|\cdot|))$ is a translation invariant proper metric space. From $L(t)/t \to \infty$, it follows that the geometric boundary $\partial(\mathbb{R}, L(|\cdot|)) = \{h \equiv 0\}$. K-Ledrappier then clearly implies that $l = 0$.QED.

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- Question: Does K-Ledrappier give something more than Oseledec in the case of certain subgroups of $GL_n\mathbb{R}$ by taking advantage of other invariant metrics, e.g. cone metrics?

An idea for distorted actions

- Let Γ be a finitely generated group with word metric $\|\cdot\|$ and which acts on a metric space (X, d) (e.g. a Hilbert space). Assume the action is distorted,

$$d(gx_0, x_0) = o(\|g\|)$$

e.g. $d(gx_0, x_0) \leq C \|g\|^{1/2}$. Cf. de Cornulier-Tessera-Valette.

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- Pull back the metric d to Γ via orbit Γx_0. Distortion implies $\partial(\Gamma, d) = \{h = 0\}$.
- Get that if
$$\int_\Omega d(A(\omega)x_0, x_0) d\mu < \infty$$
then
$$l = \lim_{n\to\infty} \frac{1}{n} d(Z_n x_0, x_0) = 0.$$