Independence and dichotomies in dynamics

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Measurable dynamics:

weak mixing	vs.	compactness
completely positive entropy	vs.	zero entropy

Topological dynamics:

weak mixing	vs.	nullness/zero sequence entropy
untameness	vs.	tameness
completely/uniformly positive entropy	vs.	zero entropy

Sizes of subsets of orbits along which independence occurs:

arbitrarily large finite	vs.	boundedly finite
infinite	vs.	finite
positive density	vs.	zero density

Let (X, μ) be a probability space. Two sets $A, B \subseteq X$ are *independent* if $\mu(A \cap B) = \mu(A)\mu(B)$.

Suppose that we have a μ-preserving action of a group G on X. By considering the function

$$g \mapsto \mu(gA \cap B) - \mu(gA)\mu(B)$$

on G one can develop various notions of asymptotic independence as dynamical expressions of indeterminism or randomness:

- ergodicity
- weak mixing
- mixing
- completely positive entropy

For a unitary representation $\pi : G \to \mathcal{B}(\mathcal{H})$ we define the following.

$$\textit{ergodicity:} \quad \inf_{(\lambda_g)\in \mathrm{P}(G)} \inf_{\xi,\zeta\in\Omega} \left| \sum_{g\in G} \lambda_g \langle \pi(g)\xi, \zeta\rangle \right| = 0 \text{ for all finite } \Omega \subseteq \mathcal{H}$$

$$\textit{weak mixing:} \quad \inf_{g\in G} \inf_{\xi,\zeta\in\Omega} |\langle \pi(g)\xi, \zeta\rangle| = 0 \text{ for all finite } \Omega \subseteq \mathcal{H}$$

$$\textit{mixing:} \quad \lim_{g\to\infty} |\langle \pi(g)\xi, \zeta\rangle| = 0 \text{ for all } \xi, \zeta \in \mathcal{H}$$

We then apply these notions to the action of G on (X, μ) by considering the associated representation of G on $L_2(X, \mu) \ominus \mathbb{C}\mathbf{1}$.

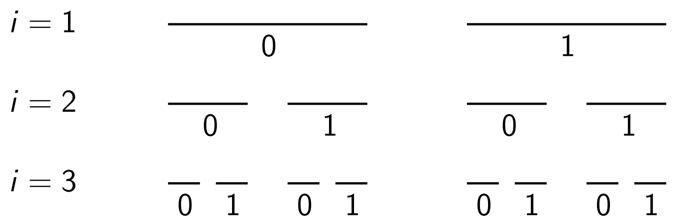
For every $\pi : G \to \mathcal{B}(\mathcal{H})$ we have an orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_{\mathrm{wm}} \oplus \mathcal{H}_{\mathrm{cpct}}$$

into G-invariant weakly mixing and compact components.

In topological dynamics, the appropriate notion of independence is the combinatorial (or set-theoretic) one.

Definition. Let X be a set. A collection $\{(A_{i,0}, A_{i,1})\}_{i=1}^{n}$ of pairs of subsets of X is said to be *independent* if $\bigcap_{i=1}^{n} A_{i,\sigma(i)} \neq \emptyset$ for every $\sigma \in \{0,1\}^{\{1,\dots,n\}}$.



Independence plays an important role in **Rosenthal's ℓ_1 theorem**:

> Every bounded sequence of functions on a set has a subsequence which either converges pointwise or is equivalent to the standard basis of ℓ_1.

Let (X, G) be a topological dynamical system. To what extent can independence be observed when we generate a family of pairs of subsets of X by applying the action of G to an initial pair (A_0, A_1)?

In the context of entropy, we are concerned with independence over subsets of orbits with positive density. The *topological entropy* of a homeomorphism $T : X \to X$ measures the asymptotic exponential growth produced by applying T iteratively to open covers:

$$\mathrm{h}_{\mathrm{top}}(T,\mathcal{U}) = \lim_{n\to\infty} \frac{1}{n} \ln N(\mathcal{U} \vee T^{-1}\mathcal{U} \vee \cdots \vee T^{-n+1}\mathcal{U}),$$
$$\mathrm{h}_{\mathrm{top}}(T) = \sup_{\mathcal{U}} \mathrm{h}_{\mathrm{top}}(T,\mathcal{U})$$

where $N(\cdot)$ denotes the minimal cardinality of a subcover.

For a two-element clopen partition $\{A_0, A_1\}$ of X the problem of positive entropy can be combinatorialized via the **Sauer-Shelah lemma**:

> For every $b > 0$ there is a $c > 0$ such that, for all $n \in \mathbb{N}$, if $S \subseteq \{0,1\}^{\{1,\dots,n\}}$ satisfies $|S| \geq e^{bn}$ then there is a $J \subseteq \{1, 2, \dots, n\}$ with $|J| \geq cn$ and $A|_J = \{0,1\}^J$.

The local theory of entropy was initiated by Blanchard about 15 years ago via the notion of entropy pair. A pair $(x, y) \in X \times X$ is an *entropy pair* if $\mathrm{h}_{\mathsf{top}}(T, \mathcal{U}) > 0$ for every open cover $\mathcal{U}$ consisting of the complements of disjoint closed neighbourhoods of x and y. Entropy tuples in X^k can be defined similarly.

Blanchard and Lacroix constructed the largest zero entropy factor of the system (X, T), called the *Pinsker factor*, by taking the smallest closed invariant equivalence relation containing all entropy pairs. The system (X, T) has

- *completely positive entropy* if every nontrivial factor has positive entropy (i.e., the Pinsker factor is trivial)
- *uniformly positive entropy* if every nondiagonal pair in $X \times X$ is an entropy pair
- *uniformly positive entropy of all orders* if for each $k \geq 2$ every nondiagonal tuple in X^k is an entropy tuple

Remarkably, every significant result to date involving entropy pairs (e.g., the product formula due to Glasner) has been obtained using measure-dynamical techniques by way of a variational principle.

Problem. Find more direct topological-combinatorial arguments.

This is one motivation for our approach to the local theory of entropy and mixing based on independence.

Definition. We call a tuple $\mathbf{x} = (x_1, \dots, x_k) \in X^k$ an *IE-tuple* if for every product neighbourhood $U_1 \times \cdots \times U_k$ of $\mathbf{x}$ the orbit of the tuple $(U_1, \dots, U_k)$ has an independent subset of positive density.

Using a local variational principle, Huang and Ye showed that entropy tuples are the same as nondiagonal IE-tuples. Inspired by work of Mendelson and Vershynin, we established a Sauer-Shelah-type coordinate density lemma that provides a combinatorial proof of this equivalence and applies in a universal way to other situations:

- entropy for actions of amenable groups
- sequence entropy for actions of arbitrary groups
- the property of tameness, which involves to the presence of ℓ_1 along infinite subsets of orbits of functions in the spirit of Rosenthal's ℓ_1 theorem

It is thus fruitful to define

IN-tuples: independence along arbitrary large finite subsets of orbits
IT-tuples: independence along infinite subsets of orbits

for the study of sequence entropy and tameness, respectively.

So the anaylsis of positive entropy production can be completely localized to the neighbourhood scale.

Question. How does the theory translate when considering locality in the dual sense at the C^*-algebra level of functions on X?

We have seen already that independence is connected with the presence of ℓ_1. The link between topological entropy and ℓ_1 structure via coordinate density was discovered by Glasner and Weiss, who used techniques from the local theory of Banach spaces to prove:

Theorem (Glasner-Weiss, 1995). If the homeomorphism $T : X \to X$ has zero entropy then so does the induced weak* homeomorphism of the space of probability measures on X.

The missing ingredient for the systematic development of the connection to Banach space geometry observed by Glasner and Weiss is Voiculescu's notion of approximation entropy:

Definition. Let A be a unital nuclear C^*-algebra. Given a finite subset $\Omega \subset A$ and $\delta > 0$, we write $\mathrm{rcp}(\Omega, \delta)$ for the infimum of d over all diagrams

$$\begin{array}{ccc} A & \xrightarrow{\ \mathrm{id}\ } & A \\ {\scriptstyle \phi}\ \searrow & & \nearrow\ {\scriptstyle \psi} \\ \text{u.c.p.} & M_d & \text{u.c.p.} \end{array}$$

which approximately commute to within δ on Ω. For an automorphism α of A we then define

$$\mathrm{ht}(\alpha, \Omega) = \sup_{\delta>0} \limsup_{n\to\infty} \frac{1}{n} \log \mathrm{rcp}(\Omega \cup \alpha\Omega \cup \cdots \cup \alpha^{n-1}\Omega, \delta),$$

$$\mathrm{ht}(\alpha) = \sup_{\Omega} \mathrm{ht}(\alpha, \Omega).$$

The *Pinsker algebra* is the invariant unital C^*-subalgebra of $C(X)$ corresponding to the Pinsker factor.

Theorem (K.-Li). For $f \in C(X)$ the following are equivalent:

(1) f is not an element of the Pinsker algebra,
(2) $\mathrm{ht}(\alpha_T, \{f\}) > 0$,
(3) there is an IE-pair $(x_1, x_2) \in X \times X$ with $f(x_1) \neq f(x_2)$,
(4) there is a positive density set $I \subset \mathbb{Z}$ such that $\{f \circ T^k\}_{k \in I}$ is equivalent to the standard basis of ℓ_1.

In particular we see that the system (X, T) has completely positive entropy if and only if the orbit of every nonscalar $f \in C(X)$ has a positive density subset equivalent to the standard basis of ℓ_1.

Question. What are the functional-analytic meanings of uniformly positive entropy and uniformly positive entropy of all orders?

What corresponds to combinatorial independence at the C^*-algebra level is tensor product independence.

Theorem (K.-Li). The system (X, T) has uniformly positive entropy of all orders (i.e., every tuple is an IE-tuple) if and only if for every finite set $\Omega \subseteq C(X)$ and $\delta > 0$ there is a finite-dimensional unital subspace $V \subseteq C(X)$ with $\Omega \subseteq_\delta V$ such that the span of the products of the subspaces in a positive density subset J of the orbit of V is canonically isomorphic to $V^{\otimes J}$.

For uniformly positive entropy we have a similar result, only now requiring the subspaces V to be 2-dimensional.

We also have analogous results for tameness and sequence entropy using IT-tuples and IN-tuples.

Problem. Can an analogous theory of combinatorial independence be developed in measurable dynamics?

It frequently happens that combinatorial independence is present but not in a robust enough way to be measure-theoretically meaningful.

Idea: Observe whether combinatorial independence occurs to the appropriate degree in orbits of tuples of subsets whenever we hide from view a small portion of the ambient space at each stage of the dynamics.

For a topological system (X, T) with T-invariant probability measure μ we can thus define *measure IE-tuples* as in the topological case but subject to this control on our observations.

What is relevant now at the algebra level is Voiculescu's von-Neumann-algebraic approximation entropy:

Definition. Let M be a hyperfinite von Neumann algebra with faithful normal state σ. Given a finite subset $\Omega \subset M$ and $\delta > 0$, we write $\mathrm{rcp}_\sigma(\Omega, \delta)$ for the infimum of d over all diagrams

$$\begin{array}{ccccc} M & & \xrightarrow{\quad \mathrm{id} \quad} & & M \\ & \searrow & & \nearrow & \\ \phi \ \text{u.c.p.} & & M_d & & \psi \ \text{u.c.p.} \end{array}$$

with $\sigma = \sigma \circ \psi \circ \varphi$ which approximately commute in the σ-norm to within δ on Ω. For an automorphism α of M we then define

$$\begin{aligned} \mathrm{hcpa}_\sigma(\alpha, \Omega) &= \sup_{\delta>0} \limsup_{n\to\infty} \frac{1}{n} \log \mathrm{rcp}_\sigma(\Omega \cup \alpha\Omega \cup \cdots \cup \alpha^{n-1}\Omega, \delta), \\ \mathrm{hcpa}_\sigma(\alpha) &= \sup_{\Omega} \mathrm{hcpa}_\sigma(\alpha, \Omega). \end{aligned}$$

Let T be a measure-preserving automorphism of the probability space (X, μ). For a finite partition $\mathcal{P}$ of X we define

$$H(\mathcal{P}) = -\sum_{P \in \mathcal{P}} \mu(P) \log \mu(P)$$

The *entropy* $h(T)$ of T measures the asymptotic exponential growth produced by applying T iteratively to finite partitions:

$$\begin{aligned} h(T, \mathcal{P}) &= \lim_{n \to \infty} \frac{1}{n} H(\mathcal{P} \vee T^{-1}\mathcal{P} \vee \cdots \vee T^{-n+1}\mathcal{P}), \\ h(T) &= \sup_{\mathcal{P}} h(T, \mathcal{P}). \end{aligned}$$

For a measure-preserving system (X, μ, T), the *Pinsker von Neumann algebra* is the invariant unital von Neumann subalgebra of $L^\infty(X, \mu)$ corresponding to the largest zero entropy factor of T.

Theorem (K.-Li). Let (X, T) be a topological system with T-invariant probability measure μ. For $f \in L^\infty(X, \mu)$ the following are equivalent:

(1) f is not an element of the Pinsker von Neumann algebra,
(2) $\mathrm{hcpa}_\mu(T, \{f\}) > 0$,
(3) every L^2 perturbation of the orbit of f contains a subset of positive density which is equivalent to the standard basis of ℓ_1,

and, in the case that $f \in C(X)$,

(4) there is a measure IE-pair $(x_1, x_2) \in X \times X$ with $f(x_1) \neq f(x_2)$.

Corollary. For a measure-preserving system (X, μ, T) the following are equivalent:

(1) (X, μ, T) has complete positive entropy,
(2) the orbit of every nonscalar $f \in L^\infty(X, \mu)$ contains a subset of positive density which is equivalent to the standard basis of ℓ_1,
(3) the induced homeomorphism of the spectrum of $L^\infty(X, \mu)$ has uniformly positive entropy of all orders.

A *joining* between two systems (Y, ν, S) and (Z, ω, T) is an invariant probability measure on $Y \times Z$ with ν and ω as marginals. The two systems are said to be *disjoint* if $\nu \times \omega$ is the only joining between them. Viewing joinings as equivariant unital positive maps $L^\infty(Y, \nu) \to L^\infty(Z, \omega)$, the above corollary gives a linear-geometric explanation for the disjointness of zero entropy systems and completely positive entropy systems.

In the context of sequence entropy we can define measure IN-tuples and a sequence version $\mathrm{hcpa}_\mu^{\mathfrak{s}}(\cdot)$ of Voiculescu's approximation entropy.

Theorem (K.-Li). Let (X, G) be a topological system with ergodic G-invariant probability measure μ. For $f \in L^\infty(X, \mu)$ the following are equivalent:

(1) f is not an element of the maximal null von Neumann algebra,
(2) $\mathrm{hcpa}_\mu^{\mathfrak{s}}(X, \{f\}) > 0$ for some sequence $\mathfrak{s}$ in G,
(3) the closure of the orbit of f in $L_2(X, \mu)$ is not compact,
(4) the orbit of every L^2 perturbation of f contains an infinite subset which is equivalent to the standard basis of ℓ_1,
(5) the orbit of every L^2 perturbation of f contains, for some $\lambda \geq 1$, arbitrarily large finite subsets which are λ-equivalent to the standard basis of ℓ_1,

and, in the case that $f \in C(X)$,

(6) there is a measure IN-pair $(x_1, x_2) \in X \times X$ with $f(x_1) \neq f(x_2)$.

Corollary. For a measure-preserving system (X, μ, G) the following are equivalent:

(1) (X, μ, G) is weakly mixing,

(2) the orbit of every $f \in L^\infty(X, \mu)$ contains, for some $\lambda \geq 1$, arbitrarily large finite subsets which are λ-equivalent to the standard basis of ℓ_1,

(3) the orbit of every nonscalar $f \in L^\infty(X, \mu)$ contains an infinite subset which is equivalent to the standard basis of ℓ_1,

(4) every tuple for the induced action of G on the spectrum of $L^\infty(X, \mu)$ is an IN-tuple,

(5) every tuple for the induced action of G on the spectrum of $L^\infty(X, \mu)$ is an IT-tuple.