On the dynamics of left orderable groups

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Les Diablerets

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Definitions

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– Bi-orderable groups have the *unique root property*: for $n \in \mathbb{N}$,

$$f^n = g^n \quad \Longrightarrow \quad f = g$$

Examples

– Torsion free Abelian groups (bi-orderable)

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– Braid groups (Dehornoy)

Some obstructions

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Question. Does there exists infinite orderable groups satisfying Kazhdan's property (T) ?

The positive cone

– Orderability is equivalent to decomposition into positive and negative cones (semigroups): $\Gamma = \Gamma_+ \cup \Gamma_- \cup \{id\}$, where $\Gamma_+ = \{f : f \succ id\}$ and $\Gamma_- = \{f : f^{-1} \in \Gamma^+\}$.

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Example. The positive cone of Dehornoy ordering on B_n is formed by the elements $\sigma \in B_n$ which may be written as a product

$$\sigma = w_0 \sigma_i^{n_1} w_1 \sigma_i^{n_2} w_2 \cdots \sigma_i^{n_k} w_k,$$

where $i \in \{1, \ldots, n-1\}$, the exponents n_r are positive, and the w_s are words on $\sigma_{i+1}^{\pm 1}, \ldots, \sigma_{n-1}^{\pm 1}$.

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General Question. Is the positive cone of an ordering finitely generated as a semigroup ?

Dynamical realizations

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– Given $\Gamma \subset \mathrm{Homeo}_+(\mathbb{R})$ we may fix a dense sequence (x_n) of points in the real line and define $f \prec g$ if and only if the first $n \geq 1$ for which $f(x_n) \neq g(x_n)$ is such that $f(x_n) < g(x_n)$ (a "dynamical lexicographic ordering").

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– Given an ordering $\preceq$ on a countable group Γ, let $t \colon \Gamma \to \mathbb{R}$ be any order preserving map (with $t(id) = 0$). Define the action of Γ on the set $t(\Gamma)$ by letting $g(t(h)) = t(gh)$. This action may be extended continuously to the whole line... (dynamical realization).

Archimedean orders and free actions

Definition. An ordering $\preceq$ on a group Γ is *Archimedean* if for every $f \neq id$ and g in Γ there exists $n \in \mathbb{N}$ such that $f^n \succ g$.

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– Dynamical realizations of Archimedean orderings are free actions (*i.e.*, no non-trivial element has fixed points).

Theorem (Hölder). Every free action by homeomorphisms of the real line is topologically semiconjugate to an action by translations.

Bi-invariant orderings and essentially free actions

– The action of a group Γ of orientation preserving homeomorphisms of the real line is *essentially free* if for each $f \neq id$ one has either

$$f(x) \geq x \text{ for every } x \in \mathbb{R}$$

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– The action of a dynamical realization of a bi-invariant ordering is essentially free.

– Every subgroup of $\mathrm{Homeo}_+(\mathbb{R})$ whose action is essentially free is bi-orderable.

Conradian orderings and resilient orbits

The Conrad property

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– Dynamical realizations of Conradian orderings have no resilient orbits. Dynamical lexicographic orderings induced from actions without resilient orbits are Conradian.

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Γ: orderable group, $\mathcal{O}(\Gamma)$: set of orderings on Γ

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– $\mathcal{O}(\Gamma)$ admits a natural topology for which a neighborhood basis for an element $\preceq$ is given by the sets of the form

$$U_{g_1,\ldots,g_n}(\preceq) = \{\leq : \quad g_i \geq id \iff g_i \succeq id\}.$$

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– The subespaces of Conradian or bi-invariant orderings are closed.

The case of finitely generated groups

$\mathcal{G} = \{f_1, \ldots, f_k\}$: finite system of generators for Γ.

–The preceding topology is induced by the metric *dist* on $\mathcal{O}(\Gamma)$ defined by

$$dist(\preceq, \leq) = 2^{-n},$$

where n is the maximum $m \geq 0$ such that the positive cones of $\preceq$ and $\leq$ coincide for elements of $\mathcal{G}$-length not larger than m.

The action of Γ on $\mathcal{O}(\Gamma)$

– Γ acts on $\mathcal{O}(\Gamma)$ by conjugacy (equivalently, by right multiplication): given an ordering $\preceq$ its image under f is the ordering $\preceq_f$ defined by

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– This action has been used by Witte-Morris to prove the following result: if Γ is amenable, then Γ is left-orderable if and only if it is locally indicable.

Infinitely generated positive cones for non isolated orders

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Proof. If $g_1, \ldots, g_n$ generate the positive cone of an ordering $\preceq$, then the only ordering in $U_{g_1,\ldots,g_n}(\preceq)$ is $\preceq$ itself, and therefore $\preceq$ is isolated in the space of orderings.

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– For a countable orderable group Γ, the fact that no ordering is isolated is equivalent to that $\mathcal{O}(\Gamma)$ is homeomorphic to the Cantor set.

Cantor sets of orderings

– $\mathcal{O}(\mathbb{Z}^n)$ for $n \geq 2$ (Sikora).

– $\mathcal{O}(\Gamma)$ for Γ finitely generated torsion free nilpotent and non rank 1 Abelian (N).

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– $\mathcal{O}(B_n)$ has isolated points (Dubrovina-Dubrovin). However, Dehornoy's ordering can be aproximated by its conjugates. (In particular, its orbit under the right action of B_n is homeomorphic to a Cantor set.)

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Theorem (Linnell, N). The space of orderings of every countable left-orderable group is either finite or contains an homeomorphic copy of the Cantor set.

– Left-orderable groups with finitely many orderings were classified by Tararin. A non trivial example is the Klein bottle group $\langle f, g : fgf^{-1} = g^{-1} \rangle$, which admits (precisely) four different orderings.

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– Therefore, the new ordering "lives" on $\mathbb{F}_n$. Clearly, if the topological perturbation was small then the new ordering is very close to the original one.

– On the other hand, the new ordering does not coincide with the original one if the dynamical realization is "non structurally stable" (which holds for free group actions).

Comments and questions

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Conjecture. The space of bi-orderings of a (non Abelian) countable free group is a Cantor set.