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This text presents a summary of my research activity to date. There is no unity of themes, even though most of my work is about interface phenomena, and especially surface phase transitions. I have tried to present topics in a logical, rather than chronological order. In Section 1, I make a brief description of the wetting transition in the Ising model, and describe two versions of the model in which it is possible to obtain rather precise informations on the macroscopic manifestations of this surface phase transition; this section is based on the works [13, 14, 18, 16, 17, 2, 3]. In Section 2, I concentrate mostly on effective interface models, and discuss the problems of pinning, wetting, critical prewetting and the temperature dependence of interface fluctuations in this setting; I also briefly recall some of the known results about the phenomenon of entropic repulsion. This section is based on the works [9, 12, 8, 4, 1, 19, 10]. Section 3 is devoted to an extension of the random-cluster representation to a large class of models including the Ashkin-Teller model; it is based on [15]. I describe an extension of Mermin-Wagner theorem about absence of continuous symmetry breaking in low-dimensional spin systems (as well as a counterexample) in Section 4; this is based on [11]. Finally, in Section 5, I summarize the results that have been obtained about the nonperturbative derivation of Ornstein-Zernike asymptotics in finite-range Ising models in any dimension; this is based on [5, 6, 7].

1 Macroscopic manifestations of the wetting transition

This section, as well as part of the next one, deals with the wetting transition. This is a surface phenomenon of major theoretical and practical interest, and still the object of active study by physicists, chemists, ... This phenomenon occurs each time some substance occupies the bulk of a system, while another substance (or another phase of the same substance) is favoured by the boundary. In this case a layer of the preferred substance can exist in the vicinity of the walls, and the question is to understand its behavior. It turns out that varying some external parameters, say the temperature, gives rise to a phase transition: In one regime, the thickness of the layer is microscopic, while it is mesoscopic in the other. Practically the manifestation is a transition from a situation where the wall is covered by a multitude of small droplets, to a situation where the wall is covered by a homogeneous film. In this section, results about this transition, and its macroscopic manifestations, are given for the simplest of the lattice gases: the Ising model.

Let $\Lambda_L \stackrel{\rm def}{=} \{-L,\dots,L\}^d$ and $\partial\Lambda_L \stackrel{\rm def}{=} \{x \in \Lambda_L \,|\, \exists y \in \mathbb{Z}^d \setminus \Lambda_L, |x-y| = 1\}$. A boundary condition is an element $\eta \in \{-1,1\}^{\partial\Lambda_L}$. A configuration $\sigma \in \{-1,1\}^{\Lambda_L}$ is η-compatible if $\sigma(x) = \eta(x)$ for all $x \in \partial\Lambda_L$.
The Hamiltonian in Λ_L with η-boundary conditions (b.c.) is defined by

$$H^\eta_{\Lambda_L}(\sigma) = \begin{cases} -\sum_{\langle t,t'\rangle\cap\Lambda_L\neq\emptyset} J(t,t')\,\sigma(t)\sigma(t') & \text{if } \sigma \text{ is } \eta\text{-compatible;} \\ +\infty & \text{otherwise,} \end{cases}$$

where $\langle t,t'\rangle$ denotes nearest neighbors and $J(t,t')$ are real numbers. The Gibbs measure in Λ_L with η-b.c. at inverse temperature $\beta \geq 0$ is defined by

$$\mu^\eta_{\Lambda_L}(\sigma) = \frac{1}{Z^\eta_{\Lambda_L}} \exp\{-\beta H^\eta_{\Lambda_L}(\sigma)\}\,.$$

In the special cases of $+$-b.c. ($\eta \equiv 1$), we write $\mu^+_{\Lambda_L}$, $Z^+_{\Lambda_L}$, ..., and similarly for $-$-b.c. ($\eta \equiv -1$).

Let $J(t,t') \equiv 1$; the $+$-phase is described by the measure $\mu^+ = \lim_{L\to\infty} \mu^+_{\Lambda_L}$; the $-$-phase is defined similarly.

Let $\vec{n}$ be a unit vector in $\mathbb{R}^d$, and $\mathcal{D}_{\vec{n}}$ be the hyperplane through the origin with normal $\vec{n}$. We denote by $D_{\vec{n}}$ the area of the surface $\mathcal{D}_{\vec{n}} \cap [-1,1]^d$, and define the following b.c.

$$\eta_{\vec{n}}(x) = \operatorname{sign}(x \cdot \vec{n}) ,$$

with $\operatorname{sign}(0) = 1$. Let $J(t,t') \equiv 1$. The *surface tension* in the direction $\vec{n}$ is defined by

$$\tau_\beta(\vec{n}) = - \lim_{L\to\infty} \frac{1}{L^{d-1} D_{\vec{n}}} \log \frac{Z^{\eta_{\vec{n}}}_{\Lambda_L}}{Z^+_{\Lambda_L}} .$$

Let now

$$J(t,t') = \begin{cases} h & \text{if } t_d = -L \text{ or } t'_d = -L \\ 1 & \text{otherwise,} \end{cases} \quad \text{and} \quad \eta_\pm(x) = \begin{cases} -1 & \text{if } x_d = -L, \\ 1 & \text{otherwise.} \end{cases} \tag{1}$$

The *wall free energy* is defined by

$$\tau_{\mathrm{bd}}(\beta,h) = - \lim_{L\to\infty} \frac{1}{(2L+1)^{d-1}} \log \frac{Z^{\eta_\pm}_{\Lambda_L}}{Z^+_{\Lambda_L}} .$$

Let us write $\tau^*(\beta) = \tau_\beta(\vec{e}^{\,d})$, where $\vec{e}^{\,d}$ is the unit vector in $\mathbb{R}^d$ with components $\vec{e}^{\,d}_k = \delta_{k,d}$. It was proved in [83] that $|\tau_{\mathrm{bd}}(\beta,h)| \le \tau^*(\beta)$, for all values of β and h.

The structure of the wall of a container, which is modeled in our model by the coupling constants between the spins in $\Lambda_L \setminus \partial\Lambda_L$ and those of $\partial\Lambda_L$, can have major effects on the behavior of the system, even deep inside the box, as will be shown later. What we will see is that it can even induce *surface phase transitions*, i.e. a dramatic change of behavior of the system resulting from a smooth change at the boundary. The prototypical example of such a phenomenon in our model is the *wetting transition.* Let us first recall the microscopic description of this transition. We refer to the original works [21, 83, 84] for details (see also [17]).

We set the coupling constants as in (1). Let $a = \pm 1$; we consider the following boundary conditions:

$$\eta_a(x) = \begin{cases} 1 & \text{if } x_d = -L, \\ a & \text{otherwise.} \end{cases} \tag{2}$$

We refer to h as the *boundary magnetic field* and a as the boundary condition. The sign of a does select the phase present in the bulk of the system, in the sense that $\lim_{L\to\infty} \mu^a_{\Lambda_L} = \mu^a$. Our main concern here, however, is not with the behavior of the system in the bulk, but rather on the effect of the boundary field h on the behavior near the wall. In order to do that, it is convenient to introduce *surface Gibbs states*, which are constructed as follows: Let $\Lambda'_L = \Lambda_L + L\vec{e}^{\,d}$ and $\mathbb{L} = \{x \in \mathbb{Z}^d \,|\, x_d \ge 0\}$; all definitions done for Λ_L are straightforwardly adapted to Λ'_L. The limiting measures $\mu^a_{\mathbb{L},\beta,h} = \lim_{L\to\infty} \mu^a_{\Lambda'_{L,\beta,h}}$ are called surface Gibbs states. The question here is the following: does the phase in the vicinity of the wall depend on the boundary condition a? In other words, does $\mu^a_{\mathbb{L},\beta,h}$ actually depend on a? It turns out that the answer depends on the value of h; more precisely, Fröhlich and Pfister [83, 84] proved that for any values of β and h, $|\tau_{\mathrm{bd}}(\beta,h)| \le \tau^*(\beta)$. Moreover, $|\tau_{\mathrm{bd}}(\beta,h)| = \tau^*(\beta)$ if and only

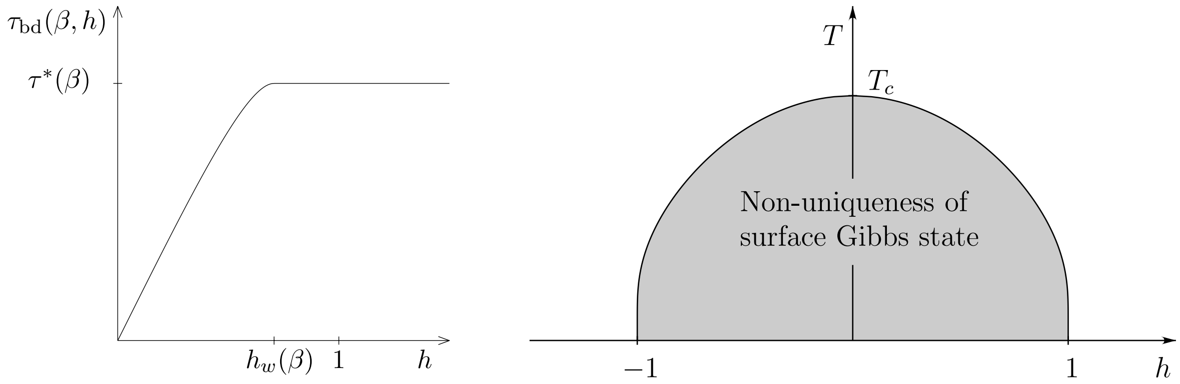

Figure 1: *Left:* The function $\tau_{\text{bd}}(\beta, h)$ for the 2-dimensional Ising model at some fixed value of $\beta > \beta_{\text{c}}$. *Right:* The wetting phase diagram for the 2-dimensional Ising model. The shaded area is the region of partial wetting.

if $\mu^{+}_{\mathbb{L},\beta,h} = \mu^{-}_{\mathbb{L},\beta,h}$; see Fig. 1 for pictures in the 2-dimensional case (the corresponding exact computations are due to [21], but can actually be deduced from [20] using duality arguments). This result provides a thermodynamical characterization of the wetting transition in terms of the surface tension and wall free energy; this characterization is known as Cahn's criterion in the physical literature.

This result can be understood heuristically. Suppose, without loss of generality, that $h \geq 0$. Then if $a = 1$, the magnetization near the wall will be positive. If $a = -1$, then there will be competition between attraction of the wall (when $h < 1$ there is an energetic gain in putting the interface along the wall) and entropic repulsion (close to the wall the phase separation line cannot fluctuate as much as far from it). Indeed, when $\tau_{\text{bd}} < \tau^*$, the attraction of the wall wins, and the phase separation line stays very close to it and visits it very often; when $\tau_{\text{bd}} = \tau^*$, entropic repulsion dominates, meaning that the phase separation line takes off and fluctuates far away from the wall (in fact at a distance of order $O(L^{1/2})$, see [21, 72, 19]). In the first situation, the phase in the bulk will extend up to the bottom wall, and consequently the magnetization near the wall will be negative. In the second situation, however, no information from the bulk can reach the bottom wall and the phase in its vicinity will have positive magnetization.

The phase transition line in the (positive h part of the) phase diagram can be parameterized by $h_{\text{w}}(\beta) = \inf\{h \in \mathbb{R} \,|\, \tau_{\text{bd}}(\beta, h) = \tau^*(\beta)\}$, as was shown in [83, 84]; the complete line can be obtained by symmetry.

Very little is known about the pathwise behavior of the system in both partial and complete wetting regimes, except in dimension 2; in the latter case, we were able to give a rather strong result about localization of the interface [14]: The probability that a given interval I along the bottom wall is not touched by the interface is smaller than $C \exp\{-(\tau^*(\beta) - \tau_{\text{bd}}(\beta, h))|I|\}$. Using concentration inequalities on the path distribution [16], this shows that the height of excursions away from the wall have exponential moments in the partial wetting regime. The results in the complete wetting regime are somehow weaker: In [19], we prove that the volume below the interface is of order $L^{3/2}$; in [72] a sketch proof of the convergence of the (suitably rescaled) interface to Brownian excursion is given, for low enough temperature and the choice $a = 1$, $h = -1$.

In the higher dimensional cases, it is proved in [83, 84] that in the partial wetting regime the probability that the interface touches the bottom wall at a given site remains bounded away from zero uniformly in L, while it vanishes in the complete wetting regime.

1.1 Grand-canonical ensemble

The wetting transition as described above cannot be observed macroscopically: in both regimes the interface lies, in the continuum limit, on the bottom wall, since it is never repelled to a macroscopic distance. In this and the next sections, we present two versions of the Ising model in which the wetting transition can be studied macroscopically; these two versions correspond, respectively, to forcing the presence of interfaces inside the system by boundary conditions, and by a volume constraint.

We start with the pinning of an interface enforced by suitable boundary conditions. To be able to see the transition, we will raise the endpoints of the interface at some macroscopic heights along the vertical walls of the box. We restrict our attention to the 2-dimensional Ising model.

The coupling constants are given by (1) with $h > 0$[1]. Let $a, b \in (-1, 1)$; the boundary condition is given by

$$\eta_{ab}(x) = \begin{cases} 1 & \text{if } x_2 = L, \\ & \text{or } x_1 = -L \text{ and } aL \le x_2 \le L, \\ & \text{or } x_1 = L \text{ and } bL \le x_2 \le L, \\ -1 & \text{otherwise.} \end{cases}$$

Let $Q = [-1, 1]^2$, and $A = (-1, a)$, $B = (1, b)$; we denote by Ω the set of all rectifiable curves inside Q with endpoints A and B. On this set we define the following (surface free energy) functional,

$$\mathcal{T}(\mathcal{C}; \beta, h) = \int_{\mathcal{C}} \tau_\beta(\underline{n}_s)\mathrm{d}s + |\mathcal{C} \cap w_Q|(\tau_{\mathrm{bd}}(\beta, h) - \tau^*(\beta)), \tag{3}$$

where $w_Q = \{x \in Q \,|\, x_2 = -1\}$ and $|\mathcal{C} \cap w_Q|$ is the length of the portion of $\mathcal{C}$ in contact with the wall w_Q.

It is not difficult to solve the corresponding thermodynamical variational problem; the result is as follows.

Let $\mathcal{D}$ be the straight line from A to B and $\mathcal{W}$ be the curve composed of three straight line segments: from A to a point $w_1 \in w_Q$, from w_1 to $w_2 \in w_Q$, and from w_2 to B; see Fig. 2. The points w_1 resp. w_2 are such that the angles between the first segment and the wall resp. between the last segment and the wall are equal to $\theta_Y \in [0, \pi/2]$, solution of the Young equation[2]

$$\cos\theta_Y\, \tau_\beta(\theta_Y) - \sin\theta_Y\, \tau'_\beta(\theta_Y) = \tau_{\mathrm{bd}}\,. \tag{4}$$

$\mathcal{W}$ is a simple curve in Q if and only if $\theta_Y \in [\arctan\frac{a+b}{2}, \pi/2)$.

We proved the following in [16]: Let θ_Y be the solution of the Young equation (4) and $\mathcal{M}_{\mathcal{T}}$ be the set of curves minimizing $\mathcal{T}$. Then

1. If $\tan\theta_Y \le \frac{a+b}{2}$, then $\mathcal{M}_{\mathcal{T}} = \{\mathcal{D}\}$.

[1] The case $h \le 0$ can be treated in exactly the same way, but is not as interesting.
[2] $\tau_\beta(\theta) \equiv \tau_\beta((\cos\theta, \sin\theta))$.

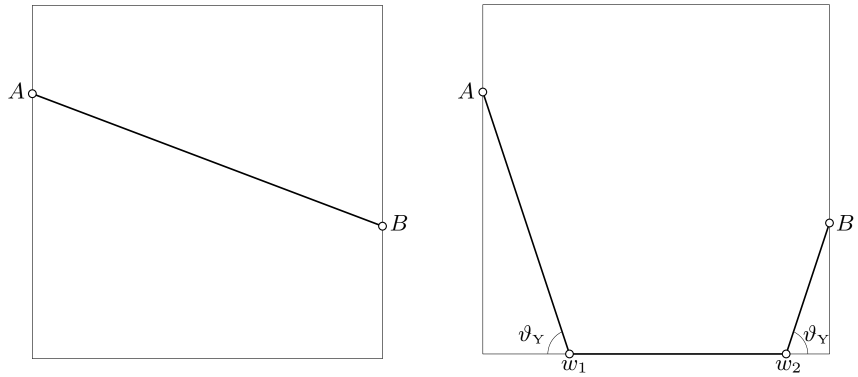

Figure 2: The two types of solutions to the variational problem. *Left:* the straight line $\mathcal{D}$. *Right:* the broken line $\mathcal{W}$.

2. If $\pi/2 > \theta_Y > \arctan(\frac{a+b}{2})$, then $\mathcal{M}_\mathcal{T} = \{\mathcal{D}\}$ if $\mathcal{T}(\mathcal{D}) < \mathcal{T}(\mathcal{W})$, $\mathcal{M}_\mathcal{T} = \{\mathcal{W}\}$ if $\mathcal{T}(\mathcal{D}) > \mathcal{T}(\mathcal{W})$ and $\mathcal{M}_\mathcal{T} = \{\mathcal{D}, \mathcal{W}\}$ if $\mathcal{T}(\mathcal{D}) = \mathcal{T}(\mathcal{W})$.

So Thermodynamics predicts that the interface of the equilibrium state should be given either by $\mathcal{D}$ or by $\mathcal{W}$, depending on the values of a, b, β and h. We show now that it is indeed possible to derive this starting from statistical mechanics. To state the result, we need to define some microscopic versions of these two curves. Let $C > 0$; we define (d_2 being the Euclidean metric)[3] [4]

$$\begin{aligned}
\mathcal{D}_L^C &= \{t \in \Lambda_L \,|\, \mathrm{d}_2(t, L\mathcal{D}) < C\sqrt{L \log L}\}\,,\\
\mathcal{W}_L^C &= \{t \in \Lambda_L \,|\, \mathrm{d}_2(t, L(\mathcal{W} \setminus w_Q)) < C\sqrt{L \log L} \text{ or } \mathrm{d}_2(t, L(\mathcal{W} \cap w_Q)) < C \log L)\}\,,\\
\partial Q_L^C &= \{t \in \Lambda_L \,|\, \mathrm{d}_2(t, \mathbb{Z}^2 \setminus \Lambda_L) < C \log L\}.
\end{aligned}$$

Let $\beta > \beta_c$ and h such that $\mathcal{T}(\mathcal{D}) \neq \mathcal{T}(\mathcal{W})$. If $\mathcal{M}_\mathcal{T} = \{\mathcal{D}\}$, then we define Λ_L^+ as the component of $\Lambda_L \setminus (\mathcal{D}_L^C \cup \partial Q_L^C)$ in contact with $+$-b.c., and Λ_L^- as the component in contact with $-$-b.c.. If $\mathcal{M}_\mathcal{T} = \{\mathcal{W}\}$, we make the corresponding definition, but this time there are two components $\Lambda_{L,1}^-$ and $\Lambda_{L,2}^-$ in contact with $-$-b.c..
The following statement follows easily from the results of [16]: Let $\beta > \beta_c$ and $h > 0$. There exist $C, c > 0$ and L_0 such that, if $\mathcal{M}_\mathcal{T} \neq \{\mathcal{D}, \mathcal{W}\}$, then for any $A \subset \Lambda_L^+$,

$$|\langle \sigma_A \rangle_{\Lambda_L}^{\eta_{ab}} - \langle \sigma_A \rangle^+| < L^{-cC}\,, \qquad \forall L \geq L_0\,.$$

The corresponding statement for $A \subset \Lambda_L^-$ (if $\mathcal{M}_\mathcal{T} = \{\mathcal{D}\}$), or $A \subset \Lambda_{L,i}^-$ (i=1,2) (if $\mathcal{M}_\mathcal{T} = \{\mathcal{W}\}$) also holds. This result shows that the set $\mathcal{D}_L^C$ or $\mathcal{W}_L^C$ corresponding to the solution of the thermodynamical variational problem does indeed play the role of a macroscopic interface, in the sense that it separates regions occupied by $+$ and $-$-phases (notice also that it converges to the corresponding solution of the variational problem as $L \to \infty$). This is therefore a precise derivation of the thermodynamical description in this case. In fact, even more can be proved, namely that for any rectifiable, simple curve $\mathcal{C}$ in Q with endpoints A and B, the probability that the interface is "close" to $\mathcal{C}$ is roughly given by $\exp\{-(\mathcal{T}(\underline{\mathcal{C}}) - \mathcal{T}^*)\, L\}$, where $\mathcal{T}^*$ is the minimum of the functional $\mathcal{T}$ on such curves. This shows that the surface free energy

[3] In fact, the sets given here are not optimal; those given in [16] are slightly better, but more complicated to describe.

[4] If $\mathcal{V} \subset \mathbb{R}^2$, then $L\mathcal{V} = \{Lx,\, x \in \mathcal{V}\}$.

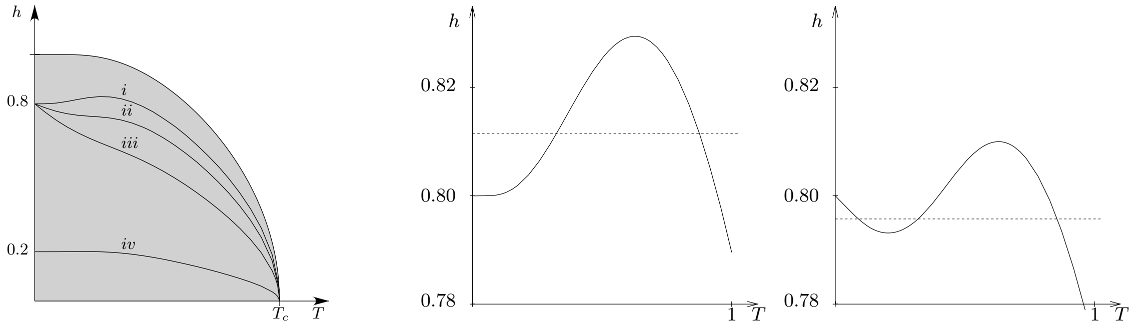

Figure 3: *Left:* The phase diagrams. The shaded area corresponds to the regime of partial wetting; it is characterized by $|\tau_{\rm bd}(\beta, h)| < \tau^*(\beta)$ or, equivalently, by non-uniqueness of the surface Gibbs state. The curves *i* to *iv* correspond to the following choices of boundary conditions: *i)* $a = b = 0.1$, *ii)* $a = 0.1, b = 0.2$, *iii)* $a = 0.1, b = 0.4$, *iv)* $a = 0.4, b = 0.4$. *Middle and right:* Enlargement of the phase transition lines corresponding to the choices $a = b = 0.1$ and $a = 0.1, b = 0.12$. Notice that both display reentrance phenomena.

is the rate-function for the large deviations of the interface, in a way completely similar to the role played by the bulk thermodynamic potentials in the case of large deviations of bulk quantities.

Since we are in the 2D Ising model, it is in fact possible to compute explicitly the corresponding phase diagrams. Figure 3 shows the phase diagrams obtained for various choices of a and b. An interesting feature is that, for some values of a and b, the system exhibits reentrance, i.e. when the temperature is increased the system goes several times from one of the phases to the other, thus displaying a non-monotonic behavior. Using self-duality of the two-dimensional Ising model, this can be reinterpreted in terms the behaviour of high-temperature correlation functions, showing that they can display non-monotonic behaviour even in fully a ferromagnetic settings; see [16] for more details.

1.2 Canonical ensemble

We describe now another situation in which the wetting transition occurring at the microscopic scale results in a transition at the macroscopic scale. We do not induce the presence of interfaces by a suitable choice of boundary conditions, but rather by a more subtle mechanism, namely fixing the total amount of magnetization. It turns out that the system reacts to such a constraint by spontaneously segregating the two phases; a crystal of one phase inside the other is thus created. There has been a large number of works dealing with equilibrium shapes of such crystals in the case of lattice gases during the last 15 years (following the much older works [110, 111]), but they were mostly neglecting boundary effects. These include, in the 2-dimensional case: [70, 116, 93, 94, 96, 32]; and in the higher-dimensional cases: [27, 28, 54, 33, 55, 2, 56, 34, 36]. The list of works dealing specifically with boundary effects and the related wetting phase transition is much shorter: [13, 14, 17, 3]. In this subsection we discuss the latter series of works.

Remark 1. *There has also been some studies of similar problems for effective interface models, which are not listed here; see [2] for a detailed bibliography.*

We start by briefly recalling the thermodynamical predictions concerning the shape of an equilibrium crystal in presence of a substrate. Let us first discuss the case of a crystal deep

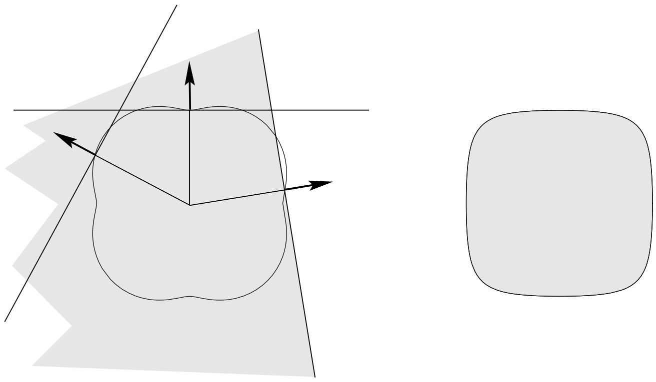

Figure 4: The Wulff construction in two dimensions: Intersecting all the half-planes (left) gives rise to the Wulff shape (right).

inside the bulk. Assuming that the crystal occupies a region $V \subset \mathbb{R}^d$, the corresponding contribution $\mathcal{W}_\beta(V)$ to the surface free energy is equal to the integral of the surface tension τ_β over the boundary ∂V of V,

$$\mathcal{W}_\beta(V) = \int_{\partial V} \tau_\beta(\vec{n}_x)\, \mathrm{d}\mathcal{H}_x^{(d-1)} ,$$

where $\mathcal{H}^{(d-1)}$ is the $(d-1)$-dimensional Hausdorff measure.

According to Thermodynamics, the shape of the equilibrium crystal is the one minimizing the surface free energy $\mathcal{W}_\beta$ at a fixed volume,

$$\mathcal{W}_\beta(V) \longrightarrow \min \qquad\qquad \text{Given: } \mathrm{vol}(V) = v\,.$$

As in the usual isoperimetric case, this variational problem is scale invariant,

$$\forall s > 0, \qquad \mathcal{W}_\beta\big(\partial(sV)\big) = s^{d-1}\mathcal{W}_\beta(\partial V).$$

Consequently, any dilatation of an optimal solution is itself optimal, and one really talks here in terms of optimal shapes. The solution to the above variational problem was first found by Wulff [124]; it is given by (any shift of) a suitably rescaled version of the convex body (see Fig. 4)

$$\mathcal{K} = \bigcap_{\vec{n}\in\mathbb{S}^{d-1}} \Big\{ x \in \mathbb{R}^d : x \;\cdot\; \vec{n} \leq \tau_\beta(\vec{n}) \Big\} \,. \tag{5}$$

The above variational problem provides a description of an equilibrium crystal deep inside the bulk. If, however, the spatial extent of the system is finite, it may happen that the boundary of the surrounding vessel exhibits a preference toward the crystal phase. In such a situation, the equilibrium state may not be given by the Wulff shape anymore, but may have the crystal attached to the boundary. We discuss briefly the simplest model of such an interaction between an equilibrium crystal and an attractive substrate. Suppose, for simplicity, that our system is contained in the half-space $H = \{x \in \mathbb{R}^d \,|\, x_d \geq 0\}$; the boundary of this half-space, the hyperplane $\mathfrak{w} \stackrel{\mathrm{def}}{=} \{x \in \mathbb{R}^d \,|\, x_d = 0\}$ represents the boundary of the vessel and is called the wall, or the substrate. We also suppose to simplify the analysis, and because these assumptions will always be satisfied, that $\tau_\beta(\vec{n}) = \tau_\beta(-\vec{n})$, and that the homogeneous extension of τ_β is convex[5].

The degree of attractiveness of the wall is modeled through the *wall free energy* $\tau_{\rm bd}(\beta, h)$, which depends on both the inverse temperature β and the "chemical structure" of the wall h, and modify the free energy functional accordingly,

$$\mathcal{W}_{\beta,h}(V) \stackrel{\rm def}{=} \mathcal{W}_\beta(V) + (\tau_{\rm bd}(\beta,h) - \tau^*_\beta)\,\mathcal{H}^{(d-1)}(\partial V \cap \mathfrak{w})\,,$$

where, as before, $\tau^*_\beta = \tau_\beta(\vec{e}^{\,d})$. The wall free energy replaces therefore the surface tension τ_β along the wall. At equilibrium, a thermodynamical stability argument shows that $\tau_{\rm bd}(\beta, h) \leq \tau^*_\beta$ (in the Ising model, this follows from the results of [83], see the beginning of the section), so that this last term is always non-positive. The new variational problem is

$$\mathcal{W}_{\beta,h}(V) \longrightarrow \min \qquad \text{Given: } V \subset H, \mathrm{vol}(V) = v\,.$$

It has first been studied in [123] and is called the Winterbottom variational problem. Let us now discuss what its solution looks like. It turns out that there are three cases to consider:

1. $\tau_{\rm bd}(\beta, h) = \tau^*_\beta$

   In this case, $\mathcal{W}_{\beta,h}(V) = \mathcal{W}_\beta(V)$ and therefore the solution is the Wulff shape associated to τ_β. The equilibrium crystal is not attached to the wall. This is the regime of *complete drying.*

2. $|\tau_{\rm bd}(\beta, h)| < \tau^*_\beta$

   Now the wall is really attracting the crystal: The crystal is attached to the wall. This is the regime of *partial wetting.* The solution of the variational problem is given by a suitably rescaled version of the following set (see Fig. 5),

   $$\mathcal{K}^{\rm w} \stackrel{\rm def}{=} \mathcal{K} \cap \{x \in \mathbb{R}^d \,|\, x_d \geq -\tau_{\rm bd}(\beta, h)\}\,,$$

   so that the volume constraint is satisfied (notice that this variational problem is still scale invariant); see [102] for a simple proof.

3. $\tau_{\rm bd}(\beta, h) = -\tau^*_\beta$

   This is a somewhat pathological case. Indeed, the solution of the variational problem is completely degenerate, the solution being unbounded. A minimizing sequence is, for example,

   $$R_n = \{x \in H \,|\, |x_k| \leq n,\, k = 1, \ldots, d-1,\, 0 \leq x_d \leq n^{1-d}\, v\}\,.$$

   As $n \to \infty$, R_n covers the whole wall with a film of vanishingly small width; the limiting value of the surface free energy functional is 0. This describes the regime of so-called *complete wetting* where the wall so strongly prefers the crystal phase that it wants to prevent any contact with the gas phase.

As can be seen from the above, the attachment of the crystal to the substrate provides another macroscopic manifestation of the wetting transition. In fact, in real situations, and as will be the case for the microscopic models studied below, we have the additional constraint

[5] In the Ising models considered below, this is a consequence of FKG inequality.

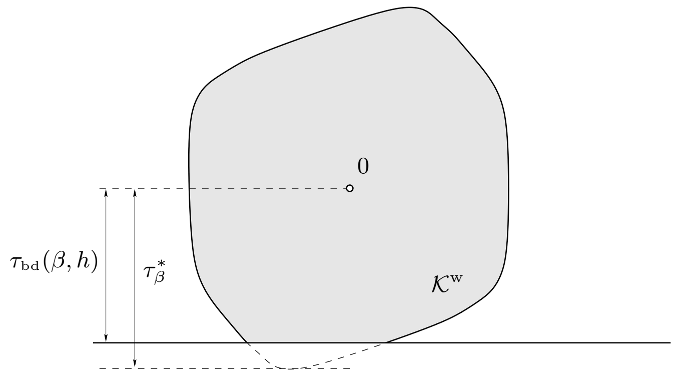

Figure 5: The Winterbottom construction: Take a Wulff shape and intersect it with a suitable half-space.

that the crystal is confined inside a box Q. We consider the case where the bottom wall w_Q of the box has a nontrivial wall free energy $\tau_{\mathrm{bd}}(\beta,h)$, while the other walls have a wall free energy equal to τ^*_β. The corresponding variational problem then takes the form

$$\mathcal{W}_{\beta,h}(V) \longrightarrow \min \qquad \text{Given: } V \subset Q, \mathrm{vol}(V) = v\,,$$

where the surface free energy functional $\mathcal{W}^Q_{\beta,h}$ is defined similarly as before

$$\mathcal{W}^Q_{\beta,h}(V) \stackrel{\text{def}}{=} \mathcal{W}_\beta(V) + (\tau_{\mathrm{bd}}(\beta,h) - \tau^*_\beta)\, \mathcal{H}^{(d-1)}(\partial V \cap w_Q)\,.$$

We refer to this variational problem as the confined Winterbottom problem. All the phenomenology presented above is still valid for small crystals (indeed, when the solution of the problem in a half-space actually fits inside Q then it is also the solution of the confined problem). However the regime of complete wetting described above cannot be observed, since it is incompatible with the confinement inside Q. The solution for large crystals is not known in general, but many cases are considered in [102].

Let us now turn to the microscopic model. The coupling constants are given[6] as before by (1), with $h \in \mathbb{R}$ (here h can be positive or negative), and we consider +-b.c.. Let $m^*(\beta) = \langle \sigma(0) \rangle^+$ be the spontaneous magnetization, and choose m such that $|m| < m^*(\beta)$. We can now define the canonical states at finite volume. Let $\epsilon > 0$. We introduce the event

$$A(m;\epsilon) \stackrel{\text{def}}{=} \{\sigma : |\sum_{t \in \Lambda_L} \sigma(t) - m|\Lambda_L|\,| \leq \epsilon\,|\Lambda_L|\}\,.$$

Remark 2. *In dimension* 2*, it is actually possible to fix* exactly *the value of the magnetization, instead of restricting it to an interval, thanks to detailed local limit estimates for the magnetization given in [96].*

The aim is to describe the typical configurations of the system under the constrained measure $\mu^+_{\Lambda_L,m,\epsilon}(\,\cdot\,) \stackrel{\text{def}}{=} \mu^+_{\Lambda_L}(\,\cdot\,|\,A(m;\epsilon))$, to which we refer as the *canonical state*, and to show that they are close, in a suitable sense, to the solution of the confined Winterbottom variational problem.

[6] Actually, the weaker results obtained in dimensions 3 and higher have been established for finite-range interactions between spins, not only nearest neighbor, and a finite-range interaction with the wall.

The formulation of the result is very different in dimension 2 and in higher dimensions. In dimension 2, it is possible to describe the typical configurations directly starting from the microscopic scale. Namely, typical configurations are described in terms of their contours, i.e. the boundaries of regions occupied by $-$ or $+$ spins. We then proved in [14] that, for any $\beta > \beta_c$ and $h \in \mathbb{R}$, up to a set of vanishing (as $L \to \infty$) $\mu^+_{\Lambda_L,m,\epsilon}$-probability,

- All contours except one have diameter smaller than $K(\beta) \log L$, where $K(\beta) < \infty$ for all $\beta > \beta_c$.
- The Hausdorff distance between the unique long contour and the boundary of (one of the solutions of) the Winterbottom variational problem confined to the box $[-L, L]^2$, and for a crystal of volume $4N^2(m^*(\beta) - m)/2m^*(\beta)$, is smaller than $CL^{3/4}\sqrt{\log L}$.

Remark 3. *The notion of convergence toward the Winterbottom shape, as used in [14], is not as strong as that. However, combining them with the local limit estimates in [96], one obtains the results stated above.*

In dimensions $d \geq 3$, the results are substantially weaker, though they still provide a suitable confirmation of thermodynamical predictions. In order to states the result, we need to introduce the notion of magnetization profiles. We partition the box Λ_N into a grid of mesh R, a sufficiently large integer to be chosen later. We then associate to any spin configuration $\sigma \in \{-1, 1\}^{\Lambda_L}$ a magnetization profile $\mathcal{M}^\sigma_L : [-1, 1]^d \to [-1, 1]$ defined by

$$\mathcal{M}^\sigma_L(x) \stackrel{\text{def}}{=} |\Delta_{R_\beta}(x)|^{-1} \sum_{x \in \Delta_{R_\beta}(x)} \sigma_x \,,$$

where $\Delta_{R_\beta}(x)$ is the cell of the grid containing the point Lx. This is just the average local magnetization around the point Lx in the configuration σ. This provides a smoothening of the microscopic configuration.

To each solution $\mathcal{V}$ of the Winterbottom variational problem confined inside $[-1, 1]^d$, and with a crystal of volume $4(m^*(\beta) - m)/2m^*(\beta)$, we associate a function $\bar{\mathcal{V}} : [-1, 1]^d \to \{-m^*(\beta), m^*(\beta)\}$ defined by

$$\bar{\mathcal{V}}(x) \stackrel{\text{def}}{=} m^*(\beta) 1_{\{x \notin \mathcal{V}\}} - m^*(\beta) 1_{\{x \in \mathcal{V}\}} \,.$$

We proved the following in [3][7]: For any $\beta > \beta_c$, any $h \in \mathbb{R}$, and any $\delta > 0$, there exists a scale $R_0(\beta, \delta)$ such that

$$\lim_{L \to \infty} \mu^+_{\Lambda_L,m,\epsilon} \left(\inf_{\mathcal{V}} \|\mathcal{M}^\cdot_L - \bar{\mathcal{V}}\|_1 < \delta \right) = 1 \,,$$

uniformly in all $L/d > R > R_0(\beta, \delta)$; the infimum is taken over all minimizers of the variational problem.

[7] Actually, we proved it for all temperatures above the so-called slab percolation threshold at which there is a single infinite-volume FK measure. This has recently been proved to be the case for all subcritical temperatures in the Ising model [35].

2 Results on effective interface models

Properties of interfaces in lattice gases in dimensions 3 and higher are usually very hard to analyze, except in some particular situations, as e.g. horizontal interfaces at low temperatures. In particular, the study of interfaces above the roughening transitions is far beyond current techniques. This is also the case of interfaces in the continuum, for which very few results have been obtained.

In order to gain some understanding of the behavior of such interfaces, it is very useful to consider simpler effective models. These are actually also the basic models studied theoretically by physicists. In these models, the interface is represented as the graph of a function from $\mathbb{Z}^d$ to $\mathbb{R}$ (or sometimes, $\mathbb{Z}$, but we'll stick to the real-valued case here). Namely, a configuration of the system is given by a function $\phi : \mathbb{Z}^d \to \mathbb{R}$, with the interpretation that ϕ_x is the height of the interface above site x. The Hamiltonian in a box $\Lambda \Subset \mathbb{Z}^d$ is given by

$$H_\Lambda(\phi) \stackrel{\text{def}}{=} \tfrac{1}{2} \sum_{x,y\in\Lambda} p(y-x) V(\phi_x - \phi_y) + \sum_{x\in\Lambda, y\notin\Lambda} p(y-x) V(\phi_x - \phi_y)\,,$$

where $p(\,\cdot\,)$ are the transition probabilities of an irreducible, symmetric ($p(x) = p(-x)$) random-walk $X.$ assumed to have finite variance, and $V : \mathbb{R} \to \mathbb{R}$ is an even function, strictly convex in the sense that there exists $c > 0$ such that

$$\forall x \in \mathbb{R} : \quad c < V''(x) < 1/c\,.$$

The probability measure in Λ, with 0-boundary conditions is then defined by

$$\mathbb{P}_\Lambda(\mathrm{d}\phi) \stackrel{\text{def}}{=} \frac{1}{Z_\Lambda} e^{-H_\Lambda(\phi)} \prod_{x\in\Lambda} \mathrm{d}\phi_x \prod_{y\notin\Lambda} \delta_0(\mathrm{d}\phi_y)\,,$$

where δ_0 is the Dirac mass at 0. When $\Lambda = \Lambda_N = \{-N, \ldots, N\}^d$, we simply write $\mathbb{P}_N$ instead of $\mathbb{P}_{\Lambda_N}$.

An important special case is the *harmonic crystal*, for which $V(x) = \frac{1}{2}x^2$. In that case, $\mathbb{P}_\Lambda$ is a Gaussian measure, and we denote it by $\mathbb{P}^\star_\Lambda$; a "$\star$" superscript will always denote the harmonic case.

A very useful tool in the analysis of the harmonic crystal is the *random walk* representation,

$$\mathrm{cov}^\star_\Lambda(\phi_x, \phi_y) = \mathrm{E}_x\Big(\sum_{n=0}^{\mathrm{T}^{\mathrm{exit}}_\Lambda - 1} 1_{\{X_n = y\}} \Big)\,, \tag{6}$$

where E_x denotes the law of the random walk $X.$ with $X_0 = x$, and $\mathrm{T}^{\mathrm{exit}}_\Lambda \stackrel{\text{def}}{=} \min\{n \,|\, X_n \notin \Lambda\}$.

We immediately get from this representation the fact that the limiting measure $\mathbb{P}_\infty \stackrel{\text{def}}{=} \lim_{N\to\infty} \mathbb{P}_N$ exists if and only if $d \geq 3$, thanks to the transience of the random walk. Notice, however, that in that case the limiting field is strongly correlated:

$$\mathrm{cov}^\star_\infty(\phi_x, \phi_y) \propto |x-y|^{2-d}\,,$$

as $|x-y| \to \infty$. These models are said to be *massless*. This refers to the fact that covariances do not decay exponentially with the distance; namely, the *mass* $m : \mathbb{S}^{d-1} \to \mathbb{R}$, defined by

$$m(x) \stackrel{\text{def}}{=} \liminf_{k\to\infty} -\frac{1}{k} \log \limsup_{N\to\infty} \mathrm{cov}_N(\phi_0, \phi_{[kx]})\,, \tag{7}$$

$[x]$ being the coordinatewise integer part of x, vanishes identically.

In dimensions 1 and 2 the limiting field does not exist, since the variance diverges. Actually, one has that $\mathrm{var}_N(\phi_0) \propto N$ in dimension 1, while $\mathrm{var}_N(\phi_0) \propto \log N$ when $d = 2$. In these last two cases, we say that the interface is *delocalized*. Such a delocalization can be also related to the continuous symmetry present in the formal Hamiltonian: A shift of all heights by the same amount leaves the Hamiltonian unchanged; delocalization is then a manifestation of the impossibility of breaking this symmetry in low dimensions, see Section 4 for more on this.

Two tools allow to extend these results to the whole class of strictly convex interactions. The first one is the Brascamp-Lieb inequality [46], which allows, among many other things, to bound variances in the general case from below by their harmonic counterparts. The reversed comparison can be obtained, e.g., using the inverse Brascamp-Lieb inequality of [68]. To compare off-diagonal correlations, one needs a more sophisticated tool, namely an extension of the above random walk representation to the case of arbitrary convex interactions. Such a generalization has been proposed in [68] and is a probabilistic reformulation of an earlier result, in the PDE context, by Helffer and Sjöstrand [90]. It works as follows: One construct a stochastic process $(\Phi(t), X(t))$ where

- $\Phi(\,\cdot\,)$ is a diffusion on $\mathbb{R}^\Lambda$ with invariant measure $\mathbb{P}_\Lambda$;
- given a trajectory $\phi(\,\cdot\,)$ of the process Φ, $X(t)$ is an, in general inhomogeneous, transient, continuous-time random walk on $\mathbb{Z}^d$ with life-time $\mathrm{T}_\Lambda^{\mathrm{exit}} \overset{\mathrm{def}}{=} \inf\{t \geq 0 \,|\, X(t) \notin \Lambda\}$, and time-dependent jump-rates

$$a(x, y; t) \overset{\mathrm{def}}{=} p(y - x) V''(\phi_x(t) - \phi_y(t))\,.$$

Denoting by $\mathcal{E}^\Lambda_{x,\phi}$ the law of $(\Phi(t), X(t))$ starting from the point $(x, \phi) \in \Lambda \times \mathbb{R}^\Lambda$, we have the following generalization of (6),

$$\mathrm{cov}_\Lambda(\phi_x, \phi_y) = \mathbb{E}_\Lambda\Big(\mathcal{E}^\Lambda_{x,\cdot} \int_0^{\mathrm{T}_\Lambda^{\mathrm{exit}}} 1_{\{X(s)=y\}} \mathrm{d}s\Big)\,. \tag{8}$$

Thanks to the ellipticity of the random walk $X(t)$ under the assumption of strict convexity, it is possible to obtain some off-diagonal Aronson type bounds, see [88].

2.1 Pinning by a local potential

In this section, we are interested in the effect of a local potential attracting the interface toward 0. Namely, we consider the Gibbs measure $\mathbb{P}_{\Lambda,a,b}$ corresponding to the Hamiltonian with square-well potential

$$H_{\Lambda,a,b}(\phi) \overset{\mathrm{def}}{=} H_\Lambda(\phi) - b \sum_{x\in\Lambda} 1_{\{|\phi_x| \leq a\}}\,,$$

where $a, b > 0$. We will also assume that the transition probabilities $p(\,\cdot\,)$ in $H_\Lambda(\phi)$ satisfy the slightly stronger condition that there exists $\delta > 0$ such that $\sum_{x\in\mathbb{Z}^d} p(x)|x|^{2+\delta} < \infty$.

By taking the weak limit of $\mathbb{P}_{\Lambda,a,b}$ as $a \to 0$, $b \to \infty$, $2a(e^b - 1) = \eta$, with $\eta > 0$ fixed, we obtain the so-called measure with δ-pinning, which can be explicitly written as

$$\mathbb{P}_{\Lambda,\eta}(\mathrm{d}\phi) = \frac{1}{Z_{\Lambda,\eta}} \exp[-H_\Lambda(\phi)] \prod_{x\in\Lambda} (\mathrm{d}\phi_x + \eta\, \delta_0(\mathrm{d}\phi_x)) \prod_{y\notin\Lambda} \delta_0(\mathrm{d}\phi_y)\,. \tag{9}$$

Although many of the results presented below have also been established for the square-well potential, from now on we restrict our attention, for the sake of simplicity, to the δ-pinning case.

This measure is no more (formally) invariant under vertical shifts of the interface $\phi \mapsto \phi + c$, and thus delocalization is not enforced by a Mermin-Wagner type argument. One might then wonder whether there is a transition from a localized state to a delocalized state as the strength η of the δ-pinning is varied. In [9], we proved that, for *any* choice of $\eta > 0$, the interface is localized in the very strong sense that

$$\log \mathbb{P}_{\Lambda,\eta}(\phi_0 \geq T) \asymp \begin{cases} -T & (d=1) \\ -T^2/\log T & (d=2) \end{cases} \tag{10}$$

uniformly in T large enough and $\Lambda \supseteq \Lambda^0(T)$, where $\Lambda^0(T) \Subset \mathbb{Z}^d$; moreover we have shown in [12] that, if the transition probabilities $p(\,\cdot\,)$ satisfy the following condition [8],

$$\exists \alpha > 0 : \quad \sum_{x \in \mathbb{Z}^d} p(x) e^{\alpha |x|} < \infty \,, \tag{11}$$

then, in any dimensions $d \geq 1$, uniformly in $x \in \mathbb{S}^{d-1}$,

$$m_\eta(x) > 0 \,, \tag{12}$$

where the mass $m_\eta(x)$ is defined analogously to (7),

$$m_\eta(x) \stackrel{\text{def}}{=} \liminf_{k\to\infty} -\frac{1}{k} \log \limsup_{N\to\infty} \operatorname{cov}_{N,\eta}(\phi_0, \phi_{[kx]}) > 0 \,.$$

Remark 4. *Condition* (11) *is necessary. Indeed, if the transition probabilities have a subexponential decay, then one can get a lower bound on the covariance between ϕ_0 and ϕ_x, using the random-walk representation as in* (16)*, by just letting the random walk make a direct jump from* 0 *to x. This gives rise to $m_\eta \equiv 0$, for all $\eta \geq 0$.*

Remark 5. *Apart from the easy one-dimensional case, earlier works on this topic were restricted to the Harmonic setting. We briefly recall the main contributions. Exponential decay of correlations for $d \geq 3$ was established (for the square-well potential) in [50]. Concerning the much more delicate two-dimensional case, it was shown in [78] that the expectation of $|\phi_0|$ is finite for any $a, b > 0$ (their results did not extend to the δ-pinning case, because their bound did not display the correct combination of a and b); in [39], it was proved that the variance of the field is finite and that the covariances decay exponentially; the proof relied on reflection positivity and as such had some constraints in addition to the interaction being Gaussian.*

Since the interface is delocalized in the absence of pinning potential, a natural and important problem is to understand how this delocalization takes place as the pinning parameter η vanishes. In the physics literature, it is customary to analyze such a divergence by looking at the behavior of the variance and the mass (directly related, in their jargon, to the transverse and longitudinal correlation lengths).

A first result we proved in [9, 4], valid for any strictly convex interaction, is that the variance diverges logarithmically as $\eta \searrow 0$: There exists $\eta_0 > 0$ such that

$$\operatorname{var}_{\Lambda,\eta}(\phi_0) \asymp |\log \eta| \,,$$

[8]Actually, only the case of a nearest-neighbor interactions was considered, but the argument used can be easily adapted to prove the result stated here.

uniformly in $0 < \eta < \eta_0$ and $\Lambda \supseteq \Lambda^0(\eta)$, where $\Lambda^0(\eta) \Subset \mathbb{Z}^2$.

In the harmonic case, it is actually possible to obtain much more precise informations on this divergence. This is what we analyzed in [4]; our results are as follows: Let $\mathcal{Q} = (q_{ij})$ be the 2×2 matrix with entries

$$q_{ij} = \sum_{x \in \mathbb{Z}^2} x_i x_j p(x) .$$

In dimension $d = 1$, the variance has the following behavior

$$\lim_{\Lambda \nearrow \mathbb{Z}} \mathrm{var}^\star_{\Lambda,\eta}(\phi_0) = \tfrac{1}{2}\eta^{-2} + o(\eta^{-2}) ,$$

as $\eta \searrow 0$, while in dimension $d = 2$,

$$\left| \lim_{\Lambda \nearrow \mathbb{Z}^2} \mathrm{var}^\star_{\Lambda,\eta}(\phi_0) - \frac{|\log \eta|}{2\pi \sqrt{\det \mathcal{Q}}} \right| \leq C \log |\log \eta| , \tag{13}$$

uniformly in $\eta < \eta_0$. We were also able to obtain precise estimates for the mass, provided (11) holds: Uniformly in $x \in \mathcal{S}^{d-1}$ and $\eta < \eta_0$,

$$m^\star_\eta(x) \asymp \begin{cases} \eta^2/2 & (d = 1) \\ \sqrt{\eta}\, |\log \eta|^{-3/4} & (d = 2) \\ \sqrt{\eta} & (d \geq 3) \end{cases}$$

It is rather remarkable that we can not only establish the presence of logarithmic corrections in dimension 2, but actually even compute their power.

Remark 6. *The results in dimension 1 have been obtained only in the nearest neighbor case, i.e.* $p(x) = \frac{1}{2} 1_{\{|x|=1\}}$, *to benefit from the renewal structure present in this case. On the other hand, the results hold for any interaction* V *such that* $\int e^{-V(y)/2} \mathrm{d}y < \infty$, $\int y\, e^{-V(y)/2} \mathrm{d}y = 0$ *and* $\int y^2 e^{-V(y)/2} \mathrm{d}y < \infty$. *Earlier results in dimension 1 focusing on some exactly solvable cases can be found in [51, 58, 104].*

Remark 7. *In dimensions* $d \geq 2$, *the critical behavior of the mass had only been obtained previously for a Gaussian model in the mean-field regime, i.e. for a very wide and shallow pinning potential [76, 77]. In that case, the potential is chosen of the form* $U(x) = -c\left(e^{x^2/2g^2} - 1\right)$, *with* $K \log(1 + c^{-1}) < \sqrt{g}$ *for some large enough constant* K *and* $0 < c \leq 1$. *It was then proved that the mass behaved as* $\sqrt{c}/g$, *which is not very surprising since in that case the quadratic approximation* $U \simeq \frac{c}{2q^2} x^2$ *holds over a very wide range of values of* x, *and therefore the model should look like a massive Gaussian field with mass* $m = \sqrt{c}/g$. *We emphasize that even though the result is quite intuitive, the proof that the interface indeed stays mostly in the region where the quadratic approximation applies is quite technical.*

The results stated above can be understood in the following way. Expanding the product in (9), we can express the measure $\mathbb{P}_{\Lambda,\eta}$ as a mixture,

$$\mathbb{P}_{\Lambda,\eta} = \sum_{A \subseteq \Lambda} \nu_{\Lambda,\eta}(A)\, \mathbb{P}_{\Lambda \setminus A} , \tag{14}$$

of free measures on random sets $\Lambda \setminus A$ with 0-boundary conditions; in (14), $\nu_{\Lambda,\eta}(A) \stackrel{\mathrm{def}}{=} \eta^{|A|} Z_\Lambda / Z_{\Lambda,\eta}$. It should be clear at this stage that the main issue is to understand the statistical properties of the random set of *pinned sites* $\mathcal{A}$ with law $\nu_{\Lambda,\eta}$. In particular, using this representation, we see that the covariances of the field take the form

$$\operatorname{cov}_{\Lambda,\eta}(\phi_0,\phi_x) = \sum_{A\subseteq\Lambda} \nu_{\Lambda,\eta}(A)\operatorname{cov}_{\Lambda\setminus A}(\phi_0,\phi_x)\,. \tag{15}$$

In the harmonic case, this can be further simplified using the random walk representation (6):

$$\begin{aligned}\operatorname{cov}^\star_{\Lambda,\eta}(\phi_0,\phi_x) &= \mathrm{E}_0\left[\sum_{n\geq 0} 1_{\{X_n=x\}} \sum_{A\subseteq\Lambda} \nu_{\Lambda,\eta}(A)\, 1_{\{\mathrm{T}^{\mathrm{exit}}_{\Lambda\setminus A}\geq n\}}\right]\\ &= \mathrm{E}_0\left[\sum_{n\geq 0} 1_{\{X_n=x\}}\nu_{\Lambda,\eta}(\mathcal{A}\cap X_{[0,n]}=\emptyset)\right]\,,\end{aligned} \tag{16}$$

where $X_{[0,n]} \stackrel{\mathrm{def}}{=} \{X_k,\, 0\leq k\leq n\}$. Thus our problem is reduced to the study of the Green function of the random-walk X among an annealed random set of killing obstacles $\mathcal{A}$, with $\mathcal{A}$ distributed according to $\nu_{\Lambda,\eta}$.

The event $\{\mathcal{A}\cap X_{[0,n]}=\emptyset\}$ being decreasing in $\mathcal{A}$, it would be very convenient to have stochastic comparisons between $\nu_{\Lambda,\eta}$ and suitable Bernoulli site percolation processes. It turns out that this is feasible in dimensions $d\geq 3$, for any strictly convex interactions (a fact first observed by D. Ioffe): We proved in [4] that there exist constants $0<c_1<c_2<\infty$ such that for all η small enough

$$\rho_{\Lambda,c_1\eta} \preccurlyeq \nu_{\Lambda,\eta} \preccurlyeq \rho_{\Lambda,c_2\eta}\,, \tag{17}$$

where $\rho_{\Lambda,\epsilon}$ denotes the Bernoulli site percolation process on Λ with intensity ϵ, and $\mu\preccurlyeq\nu$ (μ is strongly stochastically dominated by ν) means that, for any $x\in\Lambda$ and $\mathcal{C}\subset\Lambda\setminus\{x\}$,

$$\mu(\mathcal{A}\ni x\,|\,\mathcal{A}\setminus\{x\}=\mathcal{C}) \leq \nu(\mathcal{A}\ni x\,|\,\mathcal{A}\setminus\{x\}=\mathcal{C})\,.$$

Obviously this implies that, uniformly in $\mathcal{C}\subseteq\Lambda$ and η small enough,

$$\rho_{\Lambda,c_2\eta}(\mathcal{A}\cap\mathcal{C}=\emptyset) \leq \nu_{\Lambda,\eta}(\mathcal{A}\cap\mathcal{C}=\emptyset) \leq \rho_{\Lambda,c_1\eta}(\mathcal{A}\cap\mathcal{C}=\emptyset)\,. \tag{18}$$

The situation is more delicate in dimension 2. It turns out that a strong domination as in (17) cannot be true. However, we were able to prove that the following variant of (18) still holds: There exist constants $0<c_1<c_2<\infty$ such that, uniformly in $\mathcal{C}\subseteq\Lambda$ and η small enough,

$$\rho_{\Lambda,c_2\eta|\log\eta|^{-1/2}}(\mathcal{A}\cap\mathcal{C}=\emptyset) \leq \nu_{\Lambda,\eta}(\mathcal{A}\cap\mathcal{C}=\emptyset) \leq \rho_{\Lambda,c_1\eta|\log\eta|^{-1/2}}(\mathcal{A}\cap\mathcal{C}=\emptyset)\,. \tag{19}$$

If the proof of(17) is rather simple, (19) requires an elaborate coarse-graining procedure and is quite delicate; earlier versions were given in [9, 12]. The crucial ingredient is the fact that a single pinned site is enough to localize the interface in its vicinity. More precisely, the main point is that the typical value taken by the interface at a site at distance k from a single pinned site is of the same order as if it was in the middle of a box of radius k surrounded by pinned sites. This allows us to construct configurations of pinned sites inductively, one site at a time, starting from the boundary.

Of course, once (18) and (19) are proved, it only remains to study properties of the Green function of the random walk X in an annealed Bernoulli environment of killing obstacles. Surprisingly enough, even though this type of problems has a long history (see in particular [75, 37, 121, 122]), some parts of this analysis turn out to be quite tricky. For example, in order to prove (13), we need refinements of Donsker-Varadhan estimates on the range of a random walk (actually, we had to establish a discrete version of the recent work [29] on the moderate deviations for the volume of a Wiener sausage).

Remark 8. *In the non-Gaussian case, the decomposition* (15) *obviously still holds. However the representation* (16) *becomes much more involved. Indeed, in that case the law of the random-walk* does depend *on the set of pinned points, see* (8). *This prevents us from using immediately* (18) *and* (19) *in* (15). *This was one of the major difficulties in proving* (12).

2.2 Wetting transition

Before starting the description of the few known results concerning the wetting transition of effective interface models, we briefly recall what is known about the interaction of an interface with a neutral hard wall, i.e. the phenomenon of entropic repulsion; very good reviews about this (and related) topic can be found in [38, 87].

We present only the results that have been obtained in the case of nearest neighbor (i.e. $p(x) = 1_{\{|x|=1\}}/2d$) harmonic interactions, since they are much more precise. Analogous results at the qualitative level (i.e. without explicit expression for the multiplicative constants) have been obtained for the usual class of strictly convex interactions in [67]; extension to disordered substrate and the case of several interfaces can be found in [31, 30, 118].

The presence of the hard wall at the sites of Λ_M is modeled by the positivity constraint $\Omega_{M,+} \stackrel{\text{def}}{=} \{\phi_x \geq 0, \forall x \in \Lambda_M\}$. The measure describing this process is then the conditioned measure $\mathbb{P}^\star_{N,M,+}(\,\cdot\,) \stackrel{\text{def}}{=} \mathbb{P}^\star_N(\,\cdot\,|\,\Omega_{M,+})$.

Let us start with the simpler case of dimensions $d \geq 3$. In this case, it is customary to take first the limit $N \to \infty$, thus studying the measure $\mathbb{P}^\star_{\infty,M,+} \stackrel{\text{def}}{=} \mathbb{P}^\star_\infty(\,\cdot\,|\,\Omega_{M,+})$ (the corresponding results for the measure $\mathbb{P}^\star_{N,N,+}$ can be found in [65]). We are then interested in the large M asymptotics. It is proved in [41] that

$$\lim_{M\to\infty} \sup_{x\in\Lambda_M} \left| \frac{\mathbb{E}^\star_{\infty,M,+}(\phi_x)}{\sqrt{\log M}} - \sqrt{4G} \right| = 0\,, \tag{20}$$

where $G = \mathrm{var}^\star_\infty(\phi_0)$. This shows that the field is repelled far away from the hard wall. Actually, even more is known about the repelled field: Once the new average is subtracted, it is weakly converging to the unconstrained infinite-volume field [66], which means that both fields look locally the same: There exists a sequence a_M, with $\lim_{M\to\infty} a_M/\sqrt{4G\log M} = 1$, such that

$$\mathbb{P}^\star_{\infty,M,+}(\,\cdot + a_M) \stackrel{N\to\infty}{\Longrightarrow} \mathbb{P}^\star_\infty\,.$$

A similar phenomenon also occurs when $d = 2$. In that case, it is not possible to consider the conditioned infinite-volume measure, since even the unconditioned limiting field does not exist. In that case M is chosen such that $M = \epsilon N$ with $0 < \epsilon < 1$. Then, with $g \stackrel{\text{def}}{=} \lim_{N\to\infty} \mathrm{var}^\star_N(\phi_0)/\log N$, [40]

$$\lim_{N\to\infty} \sup_{x\in\Lambda_N} \mathbb{P}^\star_{N,\epsilon N,+}\Big(\Big|\phi_x - \sqrt{4g}\log N\Big| \geq \epsilon \log N\Big) = 0\,. \tag{21}$$

The phenomenon quantitatively captured in (20) and (21) is called entropic repulsion; it had been studied earlier in [48]. We emphasize that this behavior is rather remarkable, in that it is a manifestation of the strong correlations of these massless fields. Indeed, under the same conditioning, a random field with short-range correlations (or simply think of i.i.d. gaussians) would not undergo such a global "macroscopic" shift.

The basic picture is that, because the stiffness due to the strong correlations prevents it from deforming too much globally, the field has to shift by roughly the same amount everywhere. However, on a macroscopic box of size N the unconditioned field can have local fluctuations (the *spikes*) of as large an order as $\sqrt{\log N}$ when $d \geq 3$ and $\log N$ when $d = 2$. The repulsion occurs in order to accommodate most of the downward spikes. It is the structure of these spikes that make the 2-dimensional much more difficult to handle than the higher dimensional one. Indeed in the latter case, the spikes are essentially local, and therefore nearly independent of each other. In the former case, however, spikes are much wider, and can grow on each other. This makes it necessary to carry on a full multiscale analysis of the field.

Let us now turn to the problem of the wetting transition. The phenomenology of this transition has been discussed in Section 1. A natural model for its study in the framework of effective interface models is to introduce simultaneously a hard wall and a local pinning potential. The latter models the affinity of the wall toward the two equilibrium phases separated by the interface. The basic measure hence takes the form

$$\mathbb{P}_{N,\eta,+}(\,\cdot\,) \stackrel{\text{def}}{=} \mathbb{P}_{N,\eta}(\,\cdot \,|\, \Omega_{+,N}\,)\,. \tag{22}$$

Since the presence of a hard wall has a strongly repulsive effect on the interface, while the action of a local pinning potential has a strongly localizing effect, their simultaneous presence gives rise to a delicate competition.

The first natural question is whether a non trivial transition from a localized to a delocalized regime can occur as the intensity of the pinning potential is varied. A good way to study this problem at the thermodynamic level is through the *density of pinned sites*,

$$\rho_N = |\Lambda_N|^{-1} \sum_{x \in \Lambda_N} 1_{\{\phi_x = 0\}}\,,$$

and its limit $\rho = \lim_{N\to\infty} \rho_N$; the existence of this limit follows easily from the FKG property satisfied by the set pinned sites, see [4]. As $\rho_N = |\Lambda_N|^{-1} \eta \frac{\mathrm{d}}{\mathrm{d}\eta} \log Z_{N,\eta}$, we immediately obtain that ρ is a non-decreasing function of η. Obviously $\rho = 0$ when $\eta = 0$. It is also easy to see that $\rho > 0$ for large enough η (see [43] or [8] for two elementary arguments). Therefore the following critical value is well defined:

$$\eta_{\mathrm{c}} \stackrel{\text{def}}{=} \inf\{\eta \,|\, \rho(\eta) > 0\}\,. \tag{23}$$

The basic question is therefore whether $\eta_{\mathrm{c}} > 0$. The answer turns out to depend on the dimension and, maybe more surprisingly, on the tail of the gradient interaction:

- In the case of harmonic interactions, $\eta_{\mathrm{c}} > 0$ if and only if $d \leq 2$.
- In the case of Lipschitz interactions, $\eta_{\mathrm{c}} > 0$ for all $d \geq 1$.

The “only if” part of the first statement is proved in [42], while we proved the “if” part and the second statement in [8].

Remark 9. *Of course, the existece of a phase transition when $d = 1$ has been known for a long time, see e.g. [51, 58, 104, 79], and [97] for a very detailed analysis of pathwise properties in a particular case. Actually, such a result was first established for the much more difficult 2-dimensional Ising model [21] (it was discovered, but not understood, even earlier in [20]).*

The “only if” part of the first statement is proved by establishing a lower bound on the probability of the field being positive by constructing a sufficiently large family of pinning configurations, and using a change of measure argument. It turns out to be crucial that the downward spikes responsible for the repulsion of the interface from the wall are very thin; this does not happen in the harmonic case when $d \leq 2$. This is probably not an artifact of the proof, but a real physical ingredient. If this is indeed the case, this suggests that the spikes might be "fat" in any dimensions for Lipschitz interactions. It is also interesting to note that in the course of this proof, it was necessary to know the explicit value of the multiplicative constant in the pure entropic repulsion estimates (20) and (21).

Our proof of the "if" part of the first statement follows an idea originally introduced in [59] to study related questions for the discrete SOS model. It is purely perturbative, and does not provide a really clear picture of the mechanism responsible for the depinning of the interface. On the other hand, it is quite elementary. The idea is to show that the probability that the density of pinned sites is larger than $\epsilon > 0$ must be exponentially small in $|\Lambda_N|$. This is done by mapping the set of configurations with at least that many pinned points to another set with less pinned points and exponentially larger probability. The surgery used for this mapping is roughly the following one: In order to have an injective mapping, one first need to lift all the unpinned sites by some amount $c > 0$. Then, for each pinned site in the box, one flips a fair coin to decide whether to unpin it or not. If it is unpinned, then its new height is chosen uniformly in $(0, c)$. The energetic cost for doing both the shift of the field and the unpinning of the chosen sites turns out to be much smaller than the entropy increase related to the choice of the sites to unpin, provided η is small enough. If this argument can be immediately implemented for Lipschitz interactions, there is an additional ingredient that turns out to be crucial in the harmonic case: In order to control the energetic costs, one needs to ensure that the average height at a site neighboring a pinned site is finite. In other words, does the infinite field conditionned to be positive and pinned only at the origin exist? The answer turns out to be positive only when $d \leq 2$, see [78, 87].

The results proved in [42, 8] take actually a slightly stronger form: If $d \geq 3$ and for any $\eta > 0$, it is shown in [42] that there exist $\epsilon > 0$ and $c > 0$ such that

$$\mathbb{P}_{N,\eta,+}(\rho_N \geq \epsilon) \geq 1 - e^{-c|\Lambda_N|}.$$

In any dimension, we proved in [19] for any $\eta < \eta_{\rm c}$ and any $\epsilon > 0$, there exists $c > 0$ such that

$$\mathbb{P}_{N,\eta,+}(\rho_N \geq \epsilon) \leq e^{-c|\Lambda_N|}.$$

Remark 10. *In our opinion, the results given above have an important consequence. Indeed, the usual derivation of Gaussian effective interface models from a really microscopic model (e.g. a 3-dimensional Ising model above the roughening temperature), relies mainly on a second-order expansion of the free energy along the interface. Such an approximation can be relevant only when large values of gradients have no influence on the physics of the problem. The striking difference in behavior between a harmonic interaction and a Lipschitz one (which may coincide on an arbitrarily large, but finite range of values) shows that it will be very delicate to determine the correct effective interaction (if any) to mimic the behavior of the real system. It is well known that there has been a long controversy born from disagreement between numerical experiments on lattice models and (non rigorous) predictions of renormalization group theory applied to such harmonic models, resulting in the proposal of a new type of effective models by Fisher and Jin [80]. We think that part of these problems might be related to this dependence on the tail behavior.*

This of course also shows that results obtained for the harmonic model, depending on the properties of large local fluctuations (e.g. the spikes in the entropic repulsion phenomenon) *might not be generic. It would thus be extremely interesting to have some results for models with other tail behaviors.*

Considering the very detailed pathwise description available both in the case of pure pinning and that of pure entropic repulsion, one might expect to have at least some pathwise informations for the wetting problem. This turns out to be very difficult. Actually, we give in [19] the only results known to date:

- Weak form of delocalization in the whole complete wetting regime ($d = 2$):

  $$\lim_{N\to\infty} \mathbb{E}_{N,\eta,+}(\phi_x) = \infty\,,$$

  for any $x \in \mathbb{Z}^d$ and $\eta < \eta_{\rm c}$ (and $\eta = 0$ of course).

- Strong form of delocalization in the deep complete wetting regime ($d = 2$):

  $$\mathbb{E}_{N,\eta,+}(\phi_x) \asymp \log N\,,$$

  for any $x \in \Lambda_{\lambda N}$, $0 < \lambda < 1$, and η sufficiently small.

- Strong form of localization in the deep partial wetting regime ($d = 2$):

  $$\lim_{N\to\infty} \mathbb{E}^{\star}_{N,\eta,+}(\phi_x) \leq C\,,$$

  for all $x \in \mathbb{Z}^d$ and η sufficiently large. Moreover, the mass (defined analogously as in (7)) is positive:

  $$m^{\star}_{\eta,+}(x) > 0\,,$$

  for all $x \in \mathbb{S}^1$, provided η is taken large enough.

As can be seen, the results are still very limited. Particularly annoying is the absence of pathwise localization results in the full partial wetting regime (even the *deep* partial wetting regime might be nontrivial when $d \geq 3$).

Finally, a long-term goal would be to study the pathwise delocalization as $\eta \searrow \eta_{\rm c}$. Of course, this is very far from what can be done today. Actually, if it is easy to prove (using FKG inequalities to compare wetting with pure pinning) that the transition is second order (i.e. the height really diverges as the transition point is approached, instead of " finite and jumping to infinity at the transition) when $d \geq 3$ in the harmonic case [19], even this is still open when $d = 2$.

2.3 Critical prewetting

The wetting transition occurs exactly on the phase coexistence line: Both the phase occupying the bulk of the system and the phase occupying its boundary (in the complete wetting regime) are thermodynamically stable. One might wonder what happens if one approaches the region of complete wetting from outside the phase coexistence line. Indeed, in that case the phase occupying the boundary is not thermodynamically stable anymore, and can only exist sufficiently close to the boundary thanks to the stabilizing effect of the wall. Consequently, even when the film has “infinite” thickness on the phase coexistence line, the latter must be finite away from coexistence. It is then natural to study the way this thickness diverges as the system gets closer and closer to phase coexistence.

It turns out that the actual behavior depends strongly on the nature of the system. When the system is below its roughening transition (e.g. for low temperature Ising model in dimensions $d \geq 3$, or for low temperature discrete effective interface models in dimensions $d \geq 2$), the divergence occurs through an infinite sequence of first-order phase transitions, the *layering transitions*, in which the thickness of the film increases by one microscopic unit. This phenomenon has been rigorously studied in details for the discrete SOS model in [69, 57, 103]. A proof for low temperature Ising models (when $d \geq 3$) should follow from a rather standard, though probably technically involved, Pirogov-Sinai analysis.

Of more interest here is the case of rough interfaces. For systems above their roughening temperatures (e.g. Ising model in dimension $d = 3$ between $T_{\rm r}$ and $T_{\rm c}$, the 2-dimensional Ising model at any subcritical temperature, or continuous effective interface models at any temperature), the divergence of the film thickness occurs continuously; this is the so-called *critical prewetting*.

In order to model the thermodynamical instability of the wetting layer, we consider the following modification of the measure $\mathbb{P}_{N,\eta,+}$ (with, for simplicity, the choice $p(x) = 1_{\{|x|=1\}}/2d$; see (22)),

$$\mathbb{P}_{N,\eta,\lambda,+}(\mathrm{d}\phi) \stackrel{\mathrm{def}}{=} \frac{1}{Z_{N,\eta,\lambda,+}} e^{-\lambda \sum_{x\in\Lambda_N} W(\phi_x)}\, \mathbb{P}_{N,\eta,+}(\mathrm{d}\phi)\,, \tag{24}$$

where $\lambda > 0$ measures the distance to phase coexistence, and the self-potential W should be thought of as being given by $W(x) = |x|$. Actually, since this does not make any essential differences at the level of proofs, we allow any potential W which is convex, increasing on $\mathbb{R}^+$, and satisfies the following growth condition: There exists $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that for any $\alpha > 0$, we have

$$\limsup_{x\to\infty} \frac{W(\alpha x)}{W(x)} \leq f(\alpha) < \infty\,.$$

In particular any convex increasing polynomial function is admissible, including our basic example $W(x) = x$. In the latter case, this measure gives a penalization proportional to the volume of the wetting layer (i.e. the volume between the wall and the interface modeled by ϕ), and λ can be interpreted as the difference in free energy between the unstable phase below the interface and the stable phase above.

In order to have delocalization of the interface when $\lambda \searrow 0$, it is necessary that η be such that the model at $\lambda = 0$ is in the complete wetting regime. We therefore introduce the following set of admissible values of η: $\mathsf{CW} \stackrel{\mathrm{def}}{=} \{0\} \cup \{0 < \eta < \eta_{\rm c}\}$ (see (23)).

Remark 11. *The only case previously rigorously studied in the literature is the case $d = 1$ and the interaction $V(x) = |x|$ [23]. This case turns out to be exactly solvable; in particular, the authors also obtain some explicit constants, which are out of reach in the general case. See also the heuristic discussion in [79].*

We first discuss the case $d \geq 2$. Set $\mathcal{H}_{\lambda,2} = |\log\lambda|$, and $\mathcal{H}_{\lambda,d} = |\log\lambda|^{1/2}$ when $d \geq 3$. We proved in [19] that, for all $\eta \in \mathsf{CW}$, there exist strictly positive dimension-dependent constants δ, λ_0, r and c such that for all $\lambda < \lambda_0$ and all $N > \lambda^{-r}$,

$$\mathbb{P}_{N,\eta,\lambda,+}\Big(|\Lambda_N|^{-1} \sum_{x\in\Lambda_N} \phi_x \notin \big(\delta\mathcal{H}_{\lambda,d}, \delta^{-1}\mathcal{H}_{\lambda,d}\big)\Big) \leq e^{-c\lambda W(\mathcal{H}_{\lambda,d})|\Lambda_N|}\,,$$

$$\sup_{x\in\Lambda_N} \mathbb{E}_{N,\eta,\lambda,+}(\phi_x) \leq \delta^{-1}\mathcal{H}_{\lambda,d}\,,$$

$$\inf_{x\in\epsilon\Lambda_N} \mathbb{E}_{N,\eta,\lambda,+}(\phi_x) \geq \delta\mathcal{H}_{\lambda,d}\,.$$

The lower bound on the thickness in the first statement can be strengthened replacing c by $c\delta^{-1}$ in the exponent.

Remark 12. *It is interesting to note that one can actually get local estimates in whole the complete wetting regime* CW*, even though such estimates are lacking when* $\lambda = 0$ *(see Subsection 2.2).*

Remark 13. *In the course of the proof of the above statements, it turns out to be necessary to obtain estimates on the probability that a massless field stays confined between two fixed heights. Namely, setting* $\nu(2) = 1$ *and* $\nu(d) = 1/2$ *if* $d \geq 3$*, we proved that there exist dimension-dependent constants* c_-*,* c_+*,* c *and* ℓ_0 *such that, for all* $\ell \geq \ell_0$ *and all* $N \geq e^{c\ell^{1/\nu(d)}}$*,*

$$e^{-e^{-c_+\ell^{1/\nu(d)}}|\Lambda_N|} \leq \mathbb{P}_N\left(|\phi_x| \leq \ell,\, \forall x \in \Lambda_N\right) \leq e^{-e^{-c_-\ell^{1/\nu(d)}}|\Lambda_N|}$$

This is a problem of independent interest. A proof limited to the harmonic case was first given in [48]. Our proof is completely different and valid for general strictly convex interactions. However, we were unable to recover their estimates for the variance and the mass of the measure conditionned by the event $\{|\phi_x| \leq \ell,\, \forall x \in \Lambda_N\}$*.*

In the one-dimensional case, substantially stronger informations can be extracted, valid for a much larger class of models. This is particularly valuable since we'd like to understand the degree of universality of the critical behavior displayed here. This is the content of the work [10]. We consider the following model: Configurations of the interface are given by $\phi \in \mathbb{Z}_+^{\Lambda_N}$, $\Lambda_N = \{-N, \dots, N\}$, i.e. we consider discrete effective interface models. Our analysis could also be performed in the case of continuous heights, but would be slightly more cumbersome. Of course, in one dimension, both continuous and discrete heights models have the same behavior. The probability measure on the set of configurations is given by

$$\mathbb{P}_{N,+,\lambda}(\phi) \stackrel{\text{def}}{=} \frac{1}{Z_{N,+,\lambda}} \exp\{-\lambda \sum_{x\in\Lambda_N} W(\phi_x)\} \prod_{x=-N}^{N-1} \pi(\phi_{x+1} - \phi_x)\,,$$

with the boundary conditions $\phi_{-N} = \phi_N = 0$. As above $\lambda > 0$ and W is a convex function satisfying the same growth condition. $\pi(\,\cdot\,)$ are transition probabilities of an aperiodic one-dimensional integer-valued random walk with zero mean and finite variance. The class of interactions we consider is therefore immensely larger than in higher dimensions. Actually, the assumptions given here on π should be optimal (in the sense that other p should give rise to models having a different critical behavior).

Remark 14. *We have refrained from introducing a pinning interaction (i.e. we have set* $\eta = 0$*), since this would not change anything in the results (as long as* $\eta \in$ CW*), but would complicate substantially the analysis. Of course, the analogue of the results given in dimensions* $d \geq 2$ *above can be proved for strictly convex interactions also in the case* $d = 1$ *with any* $\eta \in$ CW*.*

In one dimension, we can obtain a rather precise description of the full trajectory. Here I don't give the strongest results that can be obtained, but focus on a few specific ones, namely estimates on typical heights, and decay of correlations; see [10] for additional informations.

The typical width of the interface in dimensions $d \geq 2$ does not change qualitatively as the form of the self-potential W is modified. In dimension 1, this is no longer the case, and the quantity describing the typical width of the interface is $\mathcal{H}(\lambda)$, which is the unique solution of the equation

$$\lambda\mathcal{H}^2 W(2\mathcal{H}) = 1\,.$$

In particular, in the case we are mostly interested in, which is $W(x) = x$, $\mathcal{H}$ is of order to $\lambda^{-1/3}$.

The first result is analogous to what was obtained in higher dimensions: there exist strictly positive constants δ, λ_0, C, c_1 and c_2 such that for all $\lambda < \lambda_0$ and all $N > C\lambda^{-2/3}$,

$$\mathbb{P}_{N,\eta,\lambda,+}\Big(|\Lambda_N|^{-1}\sum_{x\in\Lambda_N}\phi_x \geq \delta^{-1}\mathcal{H}(\lambda)\Big) \leq \frac{1}{c_1}e^{-c_1\delta^{-1}\mathcal{H}(\lambda)^{-2}|\Lambda_N|}\,,$$

$$\mathbb{P}_{N,\eta,\lambda,+}\Big(|\Lambda_N|^{-1}\sum_{x\in\Lambda_N}\phi_x \leq \delta\mathcal{H}(\lambda)\Big) \leq \frac{1}{c_2}e^{-c_2\delta^{-2}\mathcal{H}(\lambda)^{-2}|\Lambda_N|}\,.$$

These global estimates can again be complemented with local moment estimates, exactly as above. Actually, in one dimension, we can obtain more precise estimates on the law of fluctuations on the scale $\mathcal{H}(\lambda)$: For any T large enough, and N large compared to $\mathcal{H}(\lambda)^2$, there exist $0 < c_2 < c_1 < \infty$ such that

$$\frac{1}{c_1}e^{-c_1T^{3/2}} \leq \mathbb{P}_{N,\eta,\lambda,+}\Big(\phi_0 \geq T\mathcal{H}(\lambda)\Big) \leq \frac{1}{c_2}e^{-c_2T^{3/2}}\,,$$

for all $\lambda \in (0, \lambda_0(T)]$ (uniformly on compact subsets).

Remark 15. *For a fixed, small $\lambda > 0$, this does not allow us to study arbitrarily large fluctuations on the scale $\mathcal{H}(\lambda)$. It turns out that the behaviour deeper in the tail is not universal anymore. In the case of transition probabilities $\pi(\,\cdot\,)$ with gaussian-like tails, the same result holds uniformly in all small λ, for all T large enough. However, when the tail of $\pi(\,\cdot\,)$ becomes fatter, the behaviour changes qualitatively. It is therefore rather remarkable that one can still extract some universal information (the exponent* 3/2*).*

Moreover, in one dimension, it is possible to obtain estimates on the decay of correlations, namely one can show that the covariances of ϕ_x and ϕ_y decay exponentially fast with $|x-y|$, with a correlation length of order $\mathcal{H}(\lambda)^2$.

Finally, it is worth mentioning that a weak version of some of the results obtained for 1-dimensional effective interface models can also be derived in the 2-dimensional Ising model, thus giving a partial proof that the exponent of critical wetting is also 1/3 in that case. We briefly discuss this here, even though this section is supposed to deal with effective interface models. The problem turns out to be substantially more difficult, and some aspects are still open. The main reason why this model is more difficult to study than the corresponding effective interface model is the presence of non-trivial phases above and below the interface. Indeed, if the effective weight "$\lambda\times$ the volume below the interface" is a good approximation to what happens in the Ising model for *typical* interface configurations, it becomes very bad when the volume below the interface makes a positive large deviation: in that case the system reacts by creating big droplets of the stable phase inside the film of unstable phase, thus dramatically reducing the energetic cost, which is not anymore proportional to the volume of the unstable layer, but only to its perimeter.

In [19], we have considered the following modification to the model studied in Section 1. The coupling constants are given as before by (1), with $h \in \mathbb{R}$ (here h can be positive or negative), and we consider +-boundary conditions. Moreover we add to the Hamiltonian the term

$$\lambda \sum_{x\in\Lambda_N} \sigma_x \,.$$

When $\lambda > 0$, there is a unique equilibrium state. We suppose that the system is in the complete wetting regime when $\lambda = 0$, i.e. the boundary field satisfies $h \leq -h_{\mathrm{w}}(\beta)$. We introduce two different ways of measuring the width of the film of $-$ phase along the bottom wall. The first one, which we denote by $V(\gamma)$ is the number of lattice sites below the unique open contour γ (which is really the microscopic counterpart of the macroscopic interface; see Section 1). The second one, V^-, is the number of sites of Λ_N that are $*$-connected to the bottom wall by a path of $-$ spins. Therefore, V^- is equal to $V(\gamma)$ minus the number of sites below the open contour γ that are inside a bubble of $+$ phase. We thus expect the following conjecture to hold:

Conjecture 1. *There exists a constant $c > 0$ such that*

$$\lim_{N\to\infty} \mu_{\beta,\lambda,h,N}(V^- > cV(\gamma)) = 1\,.$$

We were unfortunately unable to prove this conjecture, which involves some apparently delicate metastability issues. Our result takes the following form: For any $\beta > \beta_{\mathrm{c}}$ and $h \leq -h_{\mathrm{w}}(\beta)$, there exist $\lambda_0 > 0$, $K < \infty$ and $C > 0$ such that, for all $0 < \lambda < \lambda_0$ and $N > K\lambda^{-2/3}\,|\log\lambda|^3$,

$$\mu_{\beta,\lambda,h,N}(V(\gamma) < \lambda^{-1/3}|\Lambda_N|\,|\log\lambda|^{-2}) \leq e^{-C\lambda^{2/3}|\Lambda_N|\,|\log\lambda|^2}\,,$$
$$\mu_{\beta,\lambda,h,N}(V^- > \lambda^{-1/3}|\Lambda_N|\,|\log\lambda|^{2}) \leq e^{-C\lambda^{2/3}|\Lambda_N|\,|\log\lambda|^2}\,.$$

The logarithmic corrections in λ that appear in these statements are purely artifacts of the proof, and should not be present. Their origin is the lack of local CLT type estimates for random lines in the 2D Ising model, in the vicinity of a boundary. Such estimates are known to hold in great generality in the bulk though, as is explained in Section 5. In order to obtain results in the absence of such a local CLT, we rely on the Ornstein-Zernike behavior of two-point functions along the boundary of the system, known to hold thanks to exact computations, and suitable correlation inequalities; the cost of this procedure is the appearance of logarithmic corrections.

2.4 Temperature dependence of interface fluctuations

It is well known that horizontal interfaces of low temperatures lattice systems are rigid in dimensions $d \geq 3$. This was first established by Dobrushin [71] for the Ising model in dimension $d \geq 3$ and then extended to many other models (e.g. in [92]). This result is not expected to remain true in general in the case of tilted interfaces, but rigorous results are still essentially nonexistent. A long-standing open problem is the proof that the $(1,1,1)$-interface of the 3-dimensional Ising model is rough at all subcritical temperatures. This interface is the one obtained by imposing the following boundary conditions on the box Λ_N: all spins above or on the plane $x+y+z=0$ are set to 1, while all those below are set to -1. This is supposed to be the orientation giving rise to the "roughest" interface, and as such might be the simplest one in which to prove such a result.

An interesting result in that direction is the work of Kenyon [99, 100], an interpretation of which is the following: If one considers the uniform probability measure on all ground state interfaces compatible with these boundary conditions (i.e. all interfaces appearing in spin configurations in Λ_N minimizing the energy), then the resulting random interface has unbounded fluctuations, and actually converges to a Gaussian free field under proper normalization. This answers the question at 0 temperature. It was then often heard that the hardest part of the analysis had been done, and that a solution valid for all temperatures (or at least, all temperatures small enough) should follow from some simple monotonicity property or perturbation argument: Indeed, if the interface has already unbounded fluctuations at 0 temperature, these fluctuations can only be enhanced if the temperature is raised. Although appealing, this argument turns out to have no foundation whatsoever.

In [1], we have constructed a simple effective interface model in which we could prove that the 0 temperature fluctuations are much more important than the finite temperature ones, thus showing that such a monotonic behavior cannot be expected in general. Actually, our main point was that the interchange of the two limits $T \searrow 0$ and $N \nearrow \infty$ is by no means trivial in this context.

Our model is a standard real-valued effective interface model, as introduced above. We impose the following boundary conditions,

$$\forall i_2, \dots, i_d : \qquad \phi_{(-N,i_2,\dots,i_d)} = -N, \qquad \phi_{(N,i_2,\dots,i_d)} = N\,,$$

while we consider free (or periodic) boundary conditions on the other sides of Λ_N. The interaction is chosen to be

$$V(x) = \begin{cases} \beta|x| & \text{if } x \in [-2,2] \\ \beta x^2/2 & \text{otherwise} \end{cases}$$

This choice was made to ensure (i) enough ground-state degeneracy, and (ii) quadratic growth at infinity in order to apply techniques from [47].

We were then able to prove that, for $\beta = \infty$ and any dimension,

$$\mathrm{var}_{N,\beta=\infty}(\phi_0) = \frac{N}{2} + O(\sqrt{N})\,,$$

while, for any $0 < \beta < \infty$ there exist $C(d) < \infty$ such that

$$\mathrm{var}_{N,\beta}(\phi_0) \leq \begin{cases} C \log N & (d=2) \\ C & (d \geq 3) \end{cases}$$

Lower bounds proving that the estimates given in the second part are of correct order can be obtained using the results of [11] described in Section 4.

Similar results are expected to hold for Ising models with the corresponding boundary conditions.

3 Random-cluster representation

An essential tool in the rigorous study of Ising and Potts models is the random-cluster representation. This graphical representation unifies the treatment of all these models, and shows explicitly their close link with the Bernoulli bond percolation model, thus allowing the transfer of ideas and techniques between these models, as well as explicit comparison by means of suitable inequalities.

For example, a tool that has played an important role in recent works on phase separation is the coarse-graining introduced by A. Pisztora in [117]; it allows a detailed study of the percolative structure of typical random-cluster configurations at a sufficiently large (but still microscopic) scale. It plays a crucial role, e.g., in the non-perturbative $\mathbb{L}^1$ theory of equilibrium crystal shapes (see Subsection 1.2). A different type of applications of the random-cluster representation is to numerical simulations; in particular it is the basis of the Swendsen-Wang cluster algorithm.

In view of the usefulness of this representation, it is interesting to see whether it can be extended to a larger class of models. In [15], we introduced a large class of models, including the Ashkin-Teller (and of course the standard random cluster model), which also possess a random-cluster representation preserving the basic properties of the standard representation, namely

- FKG inequalities;
- Comparison inequalities allowing the comparison of models in the class with different parameters.
- Duality in two dimensions, a very useful property in order to prove non-perturbative results in two dimensional systems.

The Ashkin-Teller model is a four-state model that can be conveniently represented as two Ising models interacting via a quartic interaction: Let σ and τ be two Ising configurations on $\mathbb{Z}^d$, then the formal Hamiltonian of the Ashkin-Teller model can be written down as

$$H(\sigma,\tau) = -\sum_{x,y} \left(J_\sigma \sigma_x \sigma_y + J_\tau \tau_x \tau_y + J_{\sigma\tau} \sigma_x \sigma_y \tau_x \tau_y \right) ,$$

where the sum runs, say, on nearest neighbor sites (the underlying graph structure is irrelevant in this section). Here the coupling constants satisfy $J_\sigma \geq 0$, $J_\tau \geq 0$ and $\tanh J_{\sigma\tau} \geq -\tanh J_\sigma \tanh J_\tau$.

The random-cluster representation which we derived can also be expressed in terms of two coupled random-cluster models.

An interesting aspect worth pointing out is that in the non-fully ferromagnetic regime, i.e. when $J_{\sigma\tau} < 0$, the partial ordering for which the FKG property holds is not the standard one. Retranslated into the Ising spin language, this provides generalizations of Griffiths inequalities to this regime, with the direction of the inequalities reversed as compared to the usual ferromagnetic case; e.g. one has that $\langle \sigma_A \tau_B \rangle \leq \langle \sigma_A \rangle \langle \tau_B \rangle$.

Generalization proceeds then in a natural way: First, it is possible to replace Ising spins by Potts spins (possibly with different number of states in both of the interacting systems); then one can increase the number of interacting systems. The most general form of the Hamiltonian for which we proved that such a random-cluster representation is valid is then

$$H = -\sum_{x,y} \left\{ \sum_{k=1}^{N} \sum_{r_1 < \ldots < r_k} J_k^{(r_1,\ldots,r_k)} \prod_{t=1}^{k} \left(\delta_{\sigma_i^{r_t} \sigma_j^{r_t}} - 1 \right) \right\} ,$$

where N is the number of interacting systems, and $\sigma_k \in \{1, \ldots, q_k\}$ (i.e. this model describes a system with $q_1 \cdots q_N$ different spin states). Under suitable assumptions on the coupling constants, the random-cluster representation is well-defined and also preserves the basic properties of the standard representation.

Remark 16. *Simultaneously and independently, L. Chayes and J. Machta [61] have introduced an essentially identical representation for the Ashkin-Teller and related models. The case of the Ashkin-Teller model is discussed in details, and several of the properties we established are also proved in this work. Subsequently, L. Chayes and various coauthors used this representation (and some of our results not proved in [61]) to extend to the Ashkin-Teller model several classical results only proved previously for the Ising model, see e.g. [26, 62, 63, 64].*

Some other applications of this representation can be found in my PhD thesis [18].

It should also be emphasized that many results that were proved for the usual random-cluster model can be straightforwardly generalized to this more general setting. This is the case, e.g., of Pisztora's coarse graining.

4 Extension of the Mermin-Wagner theorem

A classical result of Equilibrium Statistical Physics states that a continuous symmetry cannot be broken in dimensions 1 and 2. The prototypical example is that of the 2-dimensional XY model, which is a system of $\mathbb{S}^1$-valued spins on $\mathbb{Z}^2$, with formal Hamiltonian $H(\phi) = -\sum_{x,y}\cos(\phi_x - \phi_y)$, where the sum runs over all pairs of nearest neighbor sites. This formal Hamiltonian is clearly invariant under the transformation $\forall x \in \mathbb{Z}^2 : \phi_x \mapsto \phi_x + \theta$, for any $\theta \in \mathbb{S}^1$. This system is thus invariant under the action of the group $\mathsf{SO}(2)$. The claim is then that all Gibbs states associated to this system must also be $\mathsf{SO}(2)$-invariant.

This type of results is usually associated to the names of Mermin and Wagner, who first provided the proof of such a statement in the case of the 2-dimensional quantum Heisenberg model [108] (actually they only proved that there was no spontaneous magnetization at any temperature). Since this work, a lot of effort has been made in order to reduce the underlying assumptions, and to strengthen the conclusions.

A more general framework is the following (keeping with $\mathbb{Z}^2$ as the underlying lattice for simplicity). Let G be a compact, connected Lie group. We consider a family ϕ_x, $x \in \mathbb{Z}^2$, taking value in a topological space S, and a formal Hamiltonian

$$H(\phi) = \sum_{x,y} J_{y-x} V(\phi_x - \phi_y)\,,$$

with $J_x \geq 0$ for all $x \in \mathbb{Z}^d$, which is supposed to be invariant under the action of G, $H(g\phi) = H(\phi)$ for all $g \in G$ (we suppose that there exists a continuous action of G in S). Then the following results have been known for quite some time:

1. If $V \in \mathcal{C}^2$ and the random walk on $\mathbb{Z}^2$ with transition probabilities $p_{x,y} \propto J_{y-x}$ is recurrent, then all Gibbs states are G-invariant [115].

2. If $V \in \mathcal{C}^2$ and J_x decays at least exponentially with x, then correlations decay at least algebraically at all temperatures [120].

Three main strategies have been developed in order to prove such results: The first one relies on a reduction to an effective inhomogeneous one-dimensional problem; it has the great advantage to make possible general proofs of algebraic decay of correlations, but is apparently limited to at least exponentially decaying interactions [73, 120]. The second approach is via a change of measure argument, and a relative entropy estimate; it yields very simple proofs, and allows the study of long-range interactions, but does not seem well-suited to study correlations [115, 82]. The last one relies on Bogoliubov inequalities, and has the same strengths and weaknesses as the previous one, but is in my opinion somehow less natural (from a probabilistic point of view) [107, 101, 44, 98]. An alternative proof of algebraic decay of correlation for a restricted class of models ($\mathsf{O}(N)$ models) was first given in [106] and then extended to long-range interactions in [109].

Remark 17. *The algebraic decay of correlations is known to be optimal, since Fröhlich and Spencer proved the corresponding lower bound in their famous work [85]. However, it is generally expected that this decay is exponential at all temperatures when the Lie group is nonabelian.*

Remark 18. *Concerning the absence of continuous symmetry breaking, it is known that the assumption on the recurrence of the random walk with transition probabilities $p_{x,y} \propto J_{y-x}$ is optimal, in the sense that one can construct models violating this assumption and having a spontaneously magnetized phase at low temperatures, see [44] or Theorem (20.15) in [86].*

The status of the smoothness assumption on the interaction V was, on the other hand, quite unclear. In all the proofs mentioned above, as well as in the heuristic spin-wave argument, it plays a major role. Indeed, without it, it is not clear *a priori* that massless excitations (in the form of spinwaves) can indeed be constructed, since e.g. a small difference of orientation between two neighboring spins can give rise to a large energetic cost. And indeed, in [11], we were able to prove that the Patrascioiu-Seiler model does possess non-invariant Gibbs states in two dimensions. This model is a variant of the nearest neighbors XY model, with an additional hard-core constraint: Admissible configurations must be such that $|\phi_x - \phi_y| \leq \delta$ for some $\delta \leq \pi/4$ and all pairs of nearest neighbors x and y. This shows that some amount of regularity of the interaction is indeed necessary.

The main contribution of our work [11] was to show that the smoothness assumption on V can be strongly reduced: Indeed mere continuity of V is enough[9] in order to prove both results stated above. The idea of the proof is at the same time very simple and quite natural. Heuristically, it is clear that the typical energy scale of thermal excitations of a system at temperature $T > 0$ is of order T (Boltzmann constant k_{B} being taken equal to 1 in all the present text). Therefore, fluctuations of V much smaller than T should be completely washed out by thermal fluctuations, giving rise to an effective, smoother V. Mathematically, keeping for simplicity (but in fact without lack of generality, see the paper) with the case of $\mathbb{S}^1$-valued spins, this idea is implemented as follows: Any continuous function V can be decomposed as

$$V = V_{\text{smooth}} + V_{\text{sing}} \tag{25}$$

with $V_{\text{smooth}} \in \mathcal{C}^2$, and $-\epsilon \leq V_{\text{sing}}(x) \leq 0$, $\forall x \in \mathbb{R}$, for some arbitrarily small, fixed $\epsilon > 0$. This comes from the fact that $\mathcal{C}^2$ functions are dense in the set of continuous functions, with respect to the sup-norm (essentially Weierstrass' theorem). We then use this to decompose the Boltzmann weight

$$e^{-\beta H(\phi)} = e^{-\sum_{x,y} J_{y-x} V_{\text{smooth}}(\phi_y - \phi_x)} \prod_{x,y} \Big(1 + \Big(e^{-V_{\text{sing}}(\phi_y - \phi_x)} - 1\Big)\Big) .$$

In the spirit of the decomposition used in the study of pinning and wetting described in Section 2, we then expand the product and, using the fact that $0 \leq \exp(-V_{\text{sing}}(\phi_y - \phi_x)) - 1 \leq \epsilon$, we prove that the resulting bond percolation process is stochastically dominated by an

[9] Actually, slightly less than continuity is sufficient, see [11] for details.

independent Bernoulli bond percolation process of intensity ϵ. Therefore in most of the system everything goes as if the interaction was smooth, and the bad regions are fully under control. One then shows that introducing deformed spin-waves, staying constant on the bad regions, one can still apply the main arguments of [120, 82].

Our approach also applies in the case of non-compact Lie groups, which are of particular interest to effective interface problems. The result is usually stated as implying the absence of any infinite-volume Gibbs measures [73, 82, 47]; in the case of effective interface models, this can actually be strengthened in the form of explicit lower bounds (of correct order) on interface delocalization. In this case, however, the class of interactions which we can treat is much more reduced. Indeed, they have to admit a decomposition as in (25), which is a severe restriction when the spins are not compact. Nevertheless, this makes it possible to prove such results for several important classes of interactions which could not be treated by previous techniques (e.g. the model discussed in Subsection 2.4).

5 Ornstein-Zernike theory

In 1914, the physicists Ornstein and Zernike proposed a heuristic derivation of the asymptotic behavior of density-density correlations $G(\vec{r})$ in simple fluids away from the critical point. Their motivation was the explanation of the famous phenomenon of critical opalescence. They obtained an expression of the form

$$G(\vec{r}) \simeq \frac{A_\beta}{\sqrt{|\vec{r}|^{d-1}}} \, e^{-|\vec{r}|\xi_\beta} \,, \tag{26}$$

where the value of the correlation length ξ_β depends only on the density ρ, the inverse temperature β and the spatial dimension d.

The original OZ approach hinges on the assumption that the so called direct correlation function $C(\cdot)$, which is *de facto* introduced through the renewal type relation

$$G(\vec{r}) = C(\vec{r}) + \rho \int_{\mathbb{R}^d} C(\vec{r} - \vec{r}_1) G(\vec{r}_1) \mathrm{d}\vec{r}_1, \tag{27}$$

is of an appropriately short range.

Because of the physical significance of both the conclusions and of the underlying heuristic assumptions a number of works (see e.g. [22, 114, 119, 49, 112]) were devoted to attempts to put the theory on a rigorous basis, that is to derive (26) directly from the microscopic picture of intermolecular interactions. Most of these works, however, were based on expansion/perturbation techniques and required technical low density or high/low temperature assumptions and, thereby, addressed the situation when the parameters are far away from the critical region. Since the OZ theory was, above all, intended to describe the phenomenon of critical scattering it would be of interest to devise a rigorous approach to (26) which would rely only on qualitative features of noncriticality such as, for example, finite compressibility (or finite susceptibility in the context of ferromagnetic lattice models).

Previous works in that directions have focused on rather simple models: Self-avoiding random walks in [60, 95], and Bernoulli bond percolation in [52, 53]. Results applied to the full "high-temperature" regime of these models. A crucial property of these models is the independence of the processes in disjoint slabs.

In [5] (see also [6] for a brief pedagogical description of the proof), we present a fully nonperturbative derivation of the direction dependent analog of (26) for finite-range ferromagnetic Ising model above the critical temperature in any dimension. Let $\mathbf{J} = \{J_v\}_{v\in\mathbb{Z}^d}$, be a collection of nonnegative real numbers such that $J_v = J_{-v}$ and $J_v = 0$ if $|v| > R$, where R is some finite number. The (formal) Hamiltonian is then of the form

$$H(\sigma) = -\tfrac{1}{2} \sum_{x,y\in\mathbb{Z}^d} J_{y-x}\sigma_x\sigma_y \,. \tag{28}$$

Our approach pertains to the high temperature region $\beta < \beta_c = \beta_c(\mathbf{J}, d)$, which is the set of all β such that the susceptibility $\chi_\beta = \sum_{x\in\mathbb{Z}^d}\langle\sigma_0\sigma_x\rangle_\beta$ is finite. By the Simon-Lieb argument $\chi_\beta < \infty$ implies strict exponential decay of the two-point function. Equivalently, the series

$$\chi_\beta(t) = \sum_{x\in\mathbb{Z}^d} e^{(t,x)}\langle\sigma_0\sigma_x\rangle_\beta \,, \tag{29}$$

where $(\cdot,\cdot)$ is the usual scalar product in $\mathbb{R}^d$, has a nonempty domain of convergence for each subcritical value of the inverse temperature $\beta < \beta_c$. Note that, by an important result of Aizenman, Barsky and Fernández [25], β_c is actually the usual critical temperature: The spontaneous magnetization is positive whenever $\beta > \beta_c$.

Let us use $\mathbf{K}_\beta$ to denote the domain of convergence of (29). From a purely geometric point of view the direction dependent inverse correlation length ξ_β is the support function of $\mathbf{K}_\beta$. In particular, the dependence of $\xi_\beta(n)$ on the direction $n \in \mathbb{S}^{d-1}$ is encoded in the geometry of $\partial\mathbf{K}_\beta$, in the sense that the asymptotic decay of the two-point correlation function in the high-temperature region is given by

$$\langle\sigma_0\,\sigma_x\rangle_\beta = \frac{\Psi_\beta(n_x)}{\sqrt{|x|^{(d-1)}}}\, e^{-\xi_\beta(n_x)\,|x|}\,(1 + o(1))\,, \tag{30}$$

where $n_x = x/|x| \in \mathbb{S}^{d-1}$ is the unit vector in the direction of x, and the function Ψ_β is strictly positive and analytic. Moreover, the inverse correlation length $\xi_\beta(n)$ is an analytic function of the direction $n \in \mathbb{S}^{d-1}$ in the sense that the boundary $\partial\mathbf{K}_\beta$ of $\mathbf{K}_\beta$ is analytic and strictly convex. Furthermore, the Gaussian curvature κ_β of $\partial\mathbf{K}_\beta$ is uniformly positive,

$$\bar\kappa_\beta = \min_{t\in\partial\mathbf{K}_\beta} \kappa_\beta(t) \;>\; 0.$$

Equation (30) is the rigorous counterpart to (26) in the models we consider; it has been extended in [7] to arbitrary odd-odd correlation functions, i.e. correlations of the form $\langle\sigma_A\sigma_{B+x}\rangle_\beta$ (for large $|x|$), with $|A|$ and $|B|$ odd.

Behind the proof of (30) is a local limit theorem for Ising random lines. This local limit theorem can be extended to a functional limit theorem [89] (see [7] for an expression of the variance of the limiting process in terms of the curvature κ_β); combining this with self-duality of the nearest neighbor Ising model in dimension 2, it was proved in [89] that the interface of the 2-dimensional Ising model converges under proper rescaling to Brownian bridge for all subcritical temperatures, thus generalizing the corresponding low-temperature results of [91].

The proof of (30) consists of two main parts. We start by expressing the 2-point function in terms of the random-line representation [14, 16]. One then shows, using a suitable generalization of the coarse-graining procedure introduced in [53] for Bernoulli bond percolation, that typical paths can be decomposed into irreducible pieces with nice properties. Contrarily to what happens in percolation, or for self-avoiding random walks, the pieces interact with each other, and one cannot simply use a local CLT for independent random variables to extract the desired result. It follows however from the results on the random-line representation in [14, 16] that the interaction between these irreducible pieces (thanks to their nice properties) decays exponentially fast. This allows us to express our 2-point function in terms of a suitable Ruelle operator on a countable alphabet. We then prove a general local CLT for such shifts, from which (30) follows immediately.

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