NON-GAUSSIAN SURFACE PINNED BY A WEAK POTENTIAL

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ABSTRACT. We consider a model of a two-dimensional interface of the (continuous) SOS type, with finite-range, strictly convex interactions. We prove that, under an arbitrarily weak pinning potential, the interface is localized. We consider the cases of both square well and δ potentials. Our results extend and generalize previous results for the case of nearest-neighbours Gaussian interactions in [7] and [1]. We also obtain the tail behaviour of the height distribution, which is not Gaussian.

1. INTRODUCTION

Even though the understanding of phase separation and related interfacial phenomena for two-dimensional systems such as the Ising model, has greatly improved recently, the situation for higher dimensional systems remains quite unsatisfactory. For example, even in the three-dimensional Ising model, several basic questions remain open: Existence of a roughening transition, proof that the wetting transition occurs at a non-trivial value of the boundary magnetic field (or proof of the contrary), or even instability of the $(1,1,1)$-interface. To gain some insights in these problems, it can be useful to consider simpler SOS-type, effective models for interfaces. In these models, the interface is described by a function ϕ from a set $\Lambda \subseteq \mathbb{Z}^d$ (the basis of the interface) to $\mathbb{R}$ (or $\mathbb{Z}$, but we restrict our attention to the former case) where $\phi_i \equiv \phi(i)$ represents the height of the surface above, or below, the site i; the statistical properties of the interface are described by a Gibbs measure with formal Hamiltonian of the form $\mathcal{H}(\phi) = \sum_{i,j} V_{ij}(\phi_i - \phi_j)$. Unfortunately, even these much simplified models remain rather difficult to handle, and most of the results which have been obtained are restricted to the harmonic case, where $V_{ij}(\phi_i - \phi_j) = \frac{1}{2}(\phi_i - \phi_j)^2$. It is therefore valuable to find ways to extend such results to a larger class of models, by providing arguments which are less sensitive to the particular features of the underlying interaction.

Let us consider the case $d = 2$, which describes a two-dimensional interface in a three-dimensional medium. As a consequence of the continuous symmetry $\mathcal{H}(\phi + c) = \mathcal{H}(\phi)$ ($c \in \mathbb{R}$), no infinite volume Gibbs state exists for this (formal) Hamiltonian. However, if one adds a self-potential of the form $\sum_i W(\phi_i)$ where $W(x) = mx^2$ with a positive mass m, it is well known that the situation changes drastically. Indeed, in such a case, the following holds:

(1) The infinite volume Gibbs state exists.
(2) The covariance decays exponentially.
(3) There is a spectral gap inequality (see (4)) and a log-Sobolev inequality.

In fact all the above holds uniformly in boundary conditions and size of the system. In recent years, much effort has been devoted to understanding what remains of these properties if one replaces the strictly convex potential with a non-convex $W(x)$ with (at

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least) quadratic growth at infinity; we refer [3] and references therein for a discussion of these issues.

Nevertheless, the self-potential considered in these works is always confining, i.e. has a sufficiently fast growth at infinity. Moreover, the technique used is perturbative and in particular requires a small parameter in the interaction. A delicate question is what happens if one replaces the mass term by an arbitrarily weak compactly supported self-potential $W(x) = -b\chi(|x| \leqslant a)$ with $a, b > 0$ ($\chi(\cdot)$ denoting the indicator function) or a suitably defined limit of these potentials to the δ function (see Section 2). These potentials are not confining anymore, but still break the continuous symmetry of the formal Hamiltonian.

Such a problem has already been considered in the Gaussian case. It was shown in [7, 8] that the variance of the field remains bounded when a square-well type self-potential is added, and that the covariance decays exponentially for these potentials in the "mean-field regime", both with 0-b.c.. The corresponding results for the δ limit was proved recently in [1], but required the use of periodic b.c. in order to use reflection positivity. All these results (except for the mean-field one) are valid for arbitrarily weak self-potential, i.e. any $a, b > 0$.

The aim of the present work is to extend these results to a large class of finite-range non-Gaussian interactions; we emphasize that our results are non-perturbative and only require convexity of the interaction. Our basic tool is Brascamp-Lieb inequality, see Section 3.1. In fact, our results are stronger, since we obtain precise informations on the tail of the distribution: The probability that the height of the interface above some site i is larger than T (large) is bounded from above by $\exp(-\mathcal{O}(T^2/\log T))$; this implies of course existence of all moments, including exponential ones. If the interaction has bounded second derivatives, then we also prove the corresponding lower bound, establishing in particular non-Gaussian tails for this model. This shows that localization also holds in this case, but hypercontractivity fails (because it would imply Gaussian tails). We also give an elementary argument showing that all these results do not hold if the boundary conditions grow sufficiently fast (slightly faster than linearly is enough), thereby showing that there is a strong non-uniformity in boundary conditions.

The techniques developed in the present paper can also be used to extend the result of [1] proving the existence of a massgap, i.e. exponential decay of the covariance, for the model with nearest-neighbours Gaussian interactions and periodic boundary conditions, in the presence of an arbitrarily weak δ-pinning potential, to the case of finite-range interactions and 0-b.c.. We do not state and prove such a result here since it follows easily from a combination of [1] and our central estimate, Proposition 4.1. A simple extension of this estimate, a suitable coarse-graining argument, and the use of the random walk representation for non-Gaussian covariances, are in fact sufficient to prove exponential decay of the two-point function for non-Gaussian interactions under δ-pinning [12]. This shows that point 2. above also holds for this model (at least in the limit of δ-pinning).

Our techniques rely in a crucial way on the assumption that the interaction is (uniformly strictly) convex. What happens when this assumption is relaxed is a challenging open problem. Notice that even FKG inequality is not known to hold in such a case.

We restrict our attention to dimension $2 + 1$ since it is the relevant case to describe an interface in a three-dimensional medium. It is also the most interesting one as far as pinning is concerned. Indeed, in dimensions greater than two, the situation is completely different: The mean square of the height of the interface is already finite *without* a pinning

potential. The behaviour of the two-point function is also different: without pinning potential, it has a power-law decay. However, the techniques developed here can easily be used to show that the addition of such a potential would make this decay exponential.

A very interesting, and physically important, related problem is obtained by adding the further repulsive constraint that $\phi_i \geqslant 0$, for all i. In this case, there is competition between the attraction by the pinning potential and entropic repulsion. In dimension 1, it is not difficult to see that there exists a critical value of the strength of the potential such that the interface is pinned for larger values, but delocalized for smaller ones [9] (the so-called wetting transition). In three or more dimensions, recent results for Gaussian interactions show that an arbitrarily weak pinning potential is sufficient to localize the interface [2]. The two-dimensional case remains however open. It is also not known whether localization of the interface is accompanied by positive massgap.

In Section 2, we define the models and state the main results of this paper. Proofs of these statements are given in Section 3. Our main estimate, Proposition 4.1, is proved in Section 4. Some technical estimates are given in the appendix.

2. Models and results

Let $r \in \mathbb{Z}^+$ be the *range* of the interaction. The interaction between sites i and j, $V_{i,j}(\phi_j - \phi_i)$ is supposed to satisfy the following conditions:

- **Translation invariance:** $V_{i,j} = V_{0,j-i} \equiv V_{j-i}$.
- **Finite range:** $V_k \equiv 0$ if $\|k\|_1 > r$.
- **Symmetry:** $V_k = V_{-k}$ and $V_k(x) = V_k(-x)$.
- **Smoothness:** V_k is twice continuously differentiable.
- **Irreducibility:** V_k is convex, i.e. $V_k''(x) \geqslant 0$. Moreover, there exists $c > 0$ such that the random walk on $\mathbb{Z}^2$ with transition rates

$$P_c(0, k) \triangleq \chi(V_k''(x) \geqslant c, \, \forall x \in \mathbb{R}), \qquad k \in \mathbb{Z}^2, \tag{1}$$

  is irreducible ($\chi(\mathcal{A})$ is the indicator function of the event $\mathcal{A}$).

All these conditions are natural, and standard in this kind of problems, except for the last one, which is required to use Brascamp-Lieb type inequalities [5]. These inequalities allow us to relate the variance of the field to the corresponding Gaussian quantity.

The assumption of translation invariance could be removed easily. We only left it for notational convenience.

Sometimes, we will also use the following hypothesis on the interaction:

- **Boundedness:** There exists a constant $\bar{c}$ such that $V_k''(x) < \bar{c}$, for all x and k.

The prototypical example of an interaction satisfying all of the above conditions is the

- **Gaussian interaction:** $V_k^\star(x) = c_k x^2$, with the coefficients c_k chosen in such a way as to satisfy the above assumptions.

In the following, we will distinguish quantities associated to this particular choice of interaction by adding a "$\star$" superscript.

Let $h \in \mathbb{R}$ and let $\Lambda \Subset \mathbb{Z}^2$. The Gibbs measure with h-b.c. in Λ is the probability measure on $\mathbb{R}^{\mathbb{Z}^2}$ given by

$$\mu_\Lambda^h(\mathrm{d}\phi) \triangleq \frac{1}{\mathbf{Z}_\Lambda^b} \exp\Big\{-\sum_{\langle ij \rangle_r \cap \Lambda \neq \varnothing} V_{j-i}(\phi_i - \phi_j)\Big\} \prod_{i \in \Lambda} \mathrm{d}\phi_i \prod_{j \notin \Lambda} \delta_h(\mathrm{d}\phi_j),$$

where $\langle ij \rangle_r$ is any pair of distinct sites i and j such that $\|j - i\|_1 \leqslant r$ and δ_b is the point-mass at b. Expectation value and variance with respect to μ_Λ^h are denoted by $\langle \cdot \rangle_\Lambda^h$ and $\mathrm{var}_\Lambda^h(\cdot)$.

Let a and b be two strictly positive real numbers; the potential $W : \mathbb{R} \to \mathbb{R}$ is defined by

$$W(x) \triangleq -b\chi(|x| \leqslant a).$$

The Gibbs measure with 0-b.c. on Λ and potential W is the probability measure defined by

$$\mu_\Lambda^W(\,\mathrm{d}\phi) \triangleq \frac{1}{\mathbf{Z}_\Lambda^W} \exp\{-\sum_{i\in\Lambda} W(\phi_i)\}\, \mu_\Lambda^0(\,\mathrm{d}\phi).$$

Expectation value with respect to this measure is written $\langle \cdot \rangle_\Lambda^W$.

In [1], a slightly different measure, to which we will refer as the δ-pinning, was considered. It corresponds to

$$\mu_\Lambda^J(\,\mathrm{d}\phi) = \frac{1}{\mathbf{Z}_\Lambda^J} \exp[-\sum_{\langle ij\rangle_r \subseteq \Lambda} V_{j-i}(\phi_i - \phi_j) - \sum_{\substack{\langle ij\rangle_r \\ i\in\Lambda, j\notin\Lambda}} V_{j-i}(\phi_i)] \prod_{i\in\Lambda}(\,\mathrm{d}\phi_i + e^J \delta_0(\,\mathrm{d}\phi_i)) \prod_{j\notin\Lambda} \delta_0(\,\mathrm{d}\phi_j),$$

(2)

where J is some real parameter. (In fact, they considered the Gaussian case, with periodic boundary conditions and nearest-neighbors interactions). Expectation value and variance with respect to μ_Λ^J are written $\langle \cdot \rangle_\Lambda^J$ and $\mathrm{var}_\Lambda^J(\cdot)$.

The measure in (2) can be seen as the weak limit of the measure μ_Λ^W, when $b \to \infty$ with

$$2(e^b - 1)a = e^J \tag{3}$$

(using, for example, Lebesgue's Theorem). Since the bounds given in the following results on the measure μ_Λ^W depend on the parameters a and b only through the product $e^J \equiv 2a(e^b - 1)$, they remain valid also for the case of δ-pinning.

Let $\Lambda_N \triangleq [-N, N]^2 \cap \mathbb{Z}^2$. Our first result contains an upper bound on the tail of the marginal of $\mu_{\Lambda_N}^W$ at site i; it implies readily existence of all moments, including exponential ones, for any values of b and a.

Theorem 2.1. *Consider both square-well and δ-pinning and set $2(e^b - 1)a = e^J$. There exist $C_1 = C_1(J, c, r)$ and $T_0 = T_0(J, c, r) > 2a$ such that, for all $T > T_0$ and all N,*

$$\mu_{\Lambda_N}^W(\phi_i \geqslant T) \leqslant e^{-C_1 T^2 / \log T}.$$

In particular, the variance of the field is finite, a result proven in the case of nearest-neighbors Gaussian interactions in [7] (for $\mu_{\Lambda_N}^W$) and [1] (for $\mu_{\Lambda_N}^J$)[1]. It is possible to obtain an explicit bound on this quantity for small J:

Proposition 2.1. *Consider both square-well and δ-pinning and set $2(e^b - 1)a = e^J$. There exists a constant $C_2 = C_2(J, c, r) < \infty$ such that, for any $i \in \Lambda_N$ and $\forall N$,*

$$\mathrm{var}_{\Lambda_N}^W(\phi_i) \leqslant C_2.$$

Moreover, if $e^J \sqrt{c}$ is small, then there exists $C_3 = C_3(r) > 0$ such that $C_2 \leqslant 4a^2 + \frac{C_3}{c}|\log(e^J\sqrt{c})|$.

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[1]The bound obtained in [7] involved the product ab instead of the correct scaling given in (3). Consequently, it does not imply the corresponding result on δ-pinning proven in [1].

Note that the fact that C_2 depends on J only through the product $J\sqrt{c}$ is natural, since otherwise we could improve the result by rescaling the field ϕ (see (1)).

Theorem 2.1 implies tightness and therefore existence of the infinite-volume Gibbs measure on $\mathbb{R}^{\mathbb{Z}^2}$, which we denote by μ^W. Of course, since the above results hold uniformly in N, they remain valid for μ^W. One may wonder if the estimate obtained above is just an artifact of the proof, or if it really provides the correct behavior. In the case of interactions with bounded second derivatives, it is possible to prove the following.

Theorem 2.2. *Consider both square-well and δ-pinning, set $2(e^b - 1)a = e^J$ and suppose that the boundedness assumption is verified. Then there exists a constant $C_4 = C_4(J, c, \overline{c}, r) < \infty$ such that, for all $T > 1$ and all $i \in \mathbb{Z}^2$,*

$$\mu^W(\phi_i \geqslant T) \geqslant e^{-C_4\, T^2/\log T}\,.$$

Moreover, there exists $d_0 > 0$ such that for any N and i with $\mathrm{d}(i, \mathbb{Z}^2 \setminus \Lambda_N) > d_0 T/\log T$,

$$\mu^W_{\Lambda_N}(\phi_i \geqslant T) \geqslant e^{-C_4\, T^2/\log T}\,.$$

Observe that, for finite N, a condition on the distance between the site i and the boundary of Λ_N is necessary; indeed, we easily have in the Gaussian case, using Griffiths inequality in the form stated in [7],

$$\mu^{W,\star}_{\Lambda_N}(\phi_i \geqslant T) \leqslant \mu^{0,\star}_{\Lambda_N}(\phi_i \geqslant T) = e^{-\mathcal{O}(T^2/\log \mathrm{d}(i, \mathbb{Z}^2 \setminus \Lambda_N))}\,,$$

which is consistent with our lower bound only if $\log \mathrm{d}(i, \mathbb{Z}^2 \setminus \Lambda_N) > \mathrm{const} \cdot \log T$.

Theorem 2.2 implies in particular that the tail of the one-site marginals of the infinite state are *not* Gaussian, which shows that the usual log-Sobolev inequality does not hold for this model in the infinite-volume limit, even though correlations decay exponentially (see [12]). A major difference between the situation considered here and what occurs in models with a confining self-potential is that exponential decay of correlations for the measure considered in the present paper does *not* hold uniformly in the boundary condition; indeed, it is quite clear that if the interface is lifted high enough on the boundary to preclude any visit to the neighborhood of 0, then the pinning potential does not play any role (see Proposition 2.2 below).

Since no log-Sobolev inequality hold for the pinned field, it is natural to ask whether there is a spectral gap, i.e. whether there exists a constant C, independent of N, such that

$$\mathrm{var}^J_{\Lambda_N}(F) \leqslant C \sum_{i \in \Lambda_N} \|f_i\|^2_{L^2(\mu^J_{\Lambda_N})}, \qquad \forall F \in C^1(\mathbb{R}^{\Lambda_N}) \tag{4}$$

holds. Unfortunately, we are unable to answer this question, and the best we can do in that direction is the following result, which we state without proof.

Theorem 2.3. *Consider δ-pinning. Let $q > 1$ be an integer number and p such that $\frac{1}{p} + \frac{1}{q} = 1$. There exists a constant $C_5 = C_5(J, c, r) < \infty$ such that, for any odd function $F \in C^1(\mathbb{R}^{\Lambda_N})$ with derivatives $f_i(\phi) \equiv \partial F(\phi)/\partial \phi_i$,*

$$\mathrm{var}^J_{\Lambda_N}(F) \leqslant q\, C_5 \sum_{i \in \Lambda_N} \|f_i\|^2_{L^{2p}(\mu^J_{\Lambda_N})}\,.$$

The same holds if $F(\phi) = \sum_{i \in \Lambda_N} F_i(\phi_i)$, with $F_i \in C^1(\mathbb{R})$.

The next and final result shows that the statements presented above display a strong non-uniformity w.r.t. the boundary conditions.

Proposition 2.2. *Consider the square-well potential, and suppose that the boundedness assumption is verified. Let $\zeta > 0$ and $f(N) \geqslant \zeta N \log N$; denote by $\mu_{\Lambda_N}^{f(N),W}$ the measure with $f(N)$-boundary condition and pinning potential W; let $\rho > 0$. Then, for ζ large enough, there exists $\kappa = \kappa(\zeta, c, \bar{c}, b, \rho) > 0$ such that, for any $i \neq j \in \Lambda_N$ with $\mathrm{d}(i, \Lambda_N^{\,c}) > \rho N$, $\mathrm{d}(j, \Lambda_N^{\,c}) > \rho N$, and any N large enough,*

$$\mathrm{var}_{\Lambda_N}^{f(N),W}(\phi_i) \geqslant \kappa \log N \,,$$

and, if the interaction is Gaussian,

$$\mathrm{cov}_{\Lambda_N}^{f(N),W,\star}(\phi_i, \phi_j) \geqslant \kappa \log(N/\|j - i\|_1) \,.$$

3. Proof of results

This section is devoted to the proofs of Theorem 2.1 and 2.2, and Propositions 2.1 and 2.2.

3.1. **Some techniques.** As is well-known, in the case of Gaussian interactions, it is possible to express the covariance $\mathrm{cov}_{\Lambda}^{0,\star}(\phi_i, \phi_j)$ in terms of a random walk on $\mathbb{Z}^2$ as follows: For any $\Lambda \Subset \mathbb{Z}^2$ and any $i, j \in \Lambda$ (it is possible that $i = j$), the following holds

$$\mathrm{cov}_{\Lambda}^{0,\star}(\phi_i, \phi_j) = \mathbb{E}_i\Big[\sum_{n=0}^{\tau_\Lambda} \chi(\eta_n = j)\Big] \,,$$

where $\mathbb{E}_i$ denotes expectation w.r.t. the random walk η on $\mathbb{Z}^2$ starting at i, with jump-rate $p(i, j) = 2c_{j-i}$, η_n is its position at time n and $\tau_\Lambda = \min\{n \geqslant 0 : \eta_n \notin \Lambda\}$. Notice that η is not in general the simple random walk, but, since it is finite-range, symmetric and irreducible, its has the same qualitative properties, see [14].

To be able to use the above random walk representation, one needs a tool to compare the fields μ_Λ^0 and $\mu_\Lambda^{0,\star}$. To do this, we use Brascamp-Lieb inequality in the following form: Let us introduce the following measure,

$$\mu_\Lambda^{0,t}(\,\mathrm{d}\phi) = \frac{1}{\mathbf{Z}_\Lambda^{0,t}} e^{t \sum_{i \in \Lambda} \alpha_i \phi_i} \mu_\Lambda^0(\,\mathrm{d}\phi) \,,$$

Expectation value and variance w.r.t. $\mu_{\Lambda_N}^{0,t}$ are written $\langle \cdot \rangle_{\Lambda_N}^{0,t}$ and $\mathrm{var}_{\Lambda_N}^{0,t}$. Then for any $\alpha : \Lambda \to \mathbb{R}$,

$$\mathrm{var}_\Lambda^{0,t}(\sum_{i \in \Lambda} \alpha_i \phi_i) \leqslant \frac{1}{c}\mathrm{var}_\Lambda^{0,\star}(\sum_{i \in \Lambda}\sum_{i \in \Lambda} \alpha_i \phi_i) \,, \tag{5}$$

where $\mathrm{var}_\Lambda^{0,\star}$ is the variance w.r.t. the Gaussian measure with 0-b.c. in Λ, obtained by setting $V_k(x) = x^2/2$ if $P_c(0, k) = 1$ and 0 otherwise (see (1)).

3.2. **Mean square.** We first prove Proposition 2.1.

Expectation value with respect to $\mu_{\Lambda_N}^W$ has the following convenient representation, close to the one used in [4] and [1] in the case of the δ-pinning,

$$
\begin{aligned}
\langle\,\cdot\,\rangle_{\Lambda_N}^W &= \frac{1}{\mathbf{Z}_{\Lambda_N}^W}\int \mathrm{d}\phi\, e^{-\sum_{\langle ij\rangle_r \cap \Lambda \neq \varnothing} V_{j-i}(\phi_i-\phi_j)}\prod_{j\in\Lambda_N}\Big(1+(e^b-1)\chi(|\phi_j|\leqslant a)\Big)\\
&= \sum_{A\subseteq\Lambda_N}(e^b-1)^{|A|}\frac{\mathbf{Z}_{\Lambda_N}(A)}{\mathbf{Z}_{\Lambda_N}^W}\,\langle\,\cdot\,|\,|\phi_j|\leqslant a,\,\forall j\in A\rangle_{\Lambda_N}^0\\
&= \sum_{A\subseteq\Lambda_N}\nu(A)\,\langle\,\cdot\,|\,|\phi_j|\leqslant a,\,\forall j\in A\rangle_{\Lambda_N}^0\,,
\end{aligned}
\tag{6}
$$

where

$$
\mathbf{Z}_{\Lambda_N}(A)\triangleq\int\mathrm{d}\phi\, e^{-\sum_{\langle ij\rangle_r\cap\Lambda\neq\varnothing}V_{j-i}(\phi_i-\phi_j)}\prod_{j\in A}\chi(|\phi_j|\leqslant a)
$$

and

$$
\nu(A)=(e^b-1)^{|A|}\mathbf{Z}_{\Lambda_N}(A)\big/\mathbf{Z}_{\Lambda_N}^W\,;
$$

these weights define a probability measure ν on $\{A\subseteq\Lambda_N\}$ describing the statistics of the "pinned sites", which will play the role of killing obstacles in the random-walk representation, see below.

An upper bound on the mean square height of the field is easily obtained using (6). Indeed, we can write

$$
\langle\phi_i^2\rangle_{\Lambda_N}^W = \sum_{A\subseteq\Lambda_N}\nu(A)\,\langle\phi_i^2\,|\,|\phi_j|\leqslant a,\,\forall j\in A\rangle_{\Lambda_N}^0\,.
$$

Using Lemma 5.3 and (5), we get

$$
\langle\phi_i^2\,|\,|\phi_j|\leqslant a,\,\forall j\in A\rangle_{\Lambda_N}^0\leqslant 4a^2+4\langle\phi_i^2\rangle_{A^c}^0\leqslant 4a^2+\frac{4}{c}\langle\phi_i^2\rangle_{A^c}^{0,\star}\,,
$$

where $A^c\triangleq\Lambda_N\setminus A$.

Observe that the random-walk representation of Section 2 gives

$$
\langle\phi_i^2\rangle_{A^c}^{0,\star}=\mathbb{E}_i\Big[\sum_{n=0}^{\tau_A}\chi(\eta_n=i)\Big]\,,
$$

where $\mathbb{E}_i[\,\cdot\,]$ denotes expectation with respect to the random walk starting at site i, η_n is the position of the RW at time n, and $\tau_A\triangleq\min\{n\geqslant 0:\eta_n\notin A^c\}$. This last expression can be easily bounded using a well-known result about symmetric, irreducible random walks (see e.g. **P**12.3 and **P**29.4 in [14]); we obtain

$$
\langle\phi_i^2\rangle_{A^c}^{0,\star}\leqslant\frac{\widetilde{C}}{c}\log\mathrm{d}(i,A)\,,
\tag{7}
$$

for some absolute constant $\widetilde{C}$. Let $R_{\min}$ be the smallest value of the diameter of sets B for which Proposition 4.1 applies. Since the range of the random-walk is r-connected, we can use our main estimate, Proposition 4.1, which shows that there exists $K>0$ such that ($B_R(i)$ is the ball with radius R and center i)

$$
\sum_{\substack{A\subseteq\Lambda_N\\A\cap B_R(i)=\varnothing\\A\cap B_{R+1}(i)\neq\varnothing}}\nu(A)\leqslant e^{-K\,R^2}\,,
$$

for all $R \geqslant R_{\min}$. This implies that

$$\langle \phi_i^2 \rangle_{\Lambda_N}^W \leqslant 4a^2 + \frac{C}{c} \log R_{\min} + \sum_{R \geqslant R_{\min}} \sum_{\substack{A \subseteq \Lambda_N \\ A \cap B_R(i) = \varnothing \\ A \cap B_{R+1}(i) \neq \varnothing}} \nu(A) \frac{C}{c} \log R$$

$$\leqslant 4a^2 + \frac{C}{c} \log R_{\min} + \sum_{R \geqslant R_{\min}} e^{-K\,R^2} \frac{C}{c} \log R \,.$$

This completes the proof of Proposition 2.1; the estimate on C_1 follows by taking the optimal $R_{\min}$ above.

3.3. **Tail estimate: Upper bound.** We prove now Theorem 2.1. This proof is close to the previous one. Let us first prove the upper bound. Using the representation (6), we can write

$$\langle \chi(\phi_i > T) \rangle_{\Lambda_N}^W = \sum_{A \subseteq \Lambda_N} \nu(A) \, \langle \chi(\phi_i > T) \,\big|\, |\phi_j| \leqslant a, \, \forall j \in A \rangle_{\Lambda_N}^0$$

$$= \sum_{R \geqslant 1} \sum_{\substack{A \subseteq \Lambda_N \\ A \cap B_R(i) = \varnothing \\ A \cap B_{R+1}(i) \neq \varnothing}} \nu(A) \langle \chi(\phi_i > T) \,\big|\, |\phi_j| \leqslant a, \, \forall j \in A \rangle_{\Lambda_N}^0 \,. \qquad (8)$$

Lemma 5.2 gives

$$\langle \chi(\phi_i > T) \,\big|\, |\phi_j| \leqslant a, \, \forall j \in A \rangle_{\Lambda_N}^0 \leqslant \langle \chi(\phi_i > T - a) \rangle_{A^c}^0 \,.$$

Now this probability is easily evaluated: There exists $\underline{C} > 0$ such that

$$\langle \chi(\phi_i > T - a) \rangle_{A^c}^0 \leqslant \exp(-\underline{C}\, T^2 / \log R) \,.$$

Indeed, this follows from Chebyshev's inequality, Brascamp-Lieb inequality (5) and the variance estimate (7) (notice also that $T - a > T/2$ for $T \geqslant T_0$).

Let $R_{\min}$ be large enough so that we can apply our main estimate to sets B with $\mathrm{diam}B \geqslant R_{\min}$. We get

$$\langle \chi(\phi_i > T) \rangle_{\Lambda_N}^W \leqslant e^{-\mathcal{O}(T^2)} + \sum_{R \geqslant R_{\min}} e^{-KR^2 - \underline{C}\frac{T^2}{\log R}} \,.$$

We now have to find the asymptotic behavior in T of this sum. Observe that the function $KR^2 + \underline{C}\frac{T^2}{\log R}$ is convex, with a unique minimum at R_0 solution of

$$R_0 \log R_0 = \sqrt{\underline{C}/2K}\, T \,.$$

From this, we easily get the following lower bound on R_0:

$$R_0 > \sqrt{\underline{C}/2K}\, \frac{T}{\log T} \equiv \overline{R} \,.$$

Observe that

$$1 \geqslant \frac{\overline{R}}{R_0} \geqslant 1 - \mathcal{O}(\frac{\log \log T}{\log T}) \,.$$

The required upper bound is obtained by splitting the sum in the following way:

$$\sum_{R \geqslant R_{\min}} e^{-\underline{K}R^2 - \underline{C}\frac{T^2}{\log R}} \leqslant \sum_{R=1}^{T} e^{-\underline{K}R^2 - \underline{C}\frac{T^2}{\log R}} + \sum_{R > T} e^{-\underline{K}R^2 - \underline{C}\frac{T^2}{\log R}} \,.$$

The exponential in the first sum is maximum when $R = R_0$. Therefore,

$$\sum_{R=1}^{T} e^{-\underline{K}R^2 - \underline{C}\frac{T^2}{\log R}} \leqslant T \, e^{-\underline{K}R_0^2 - \underline{C}\frac{T^2}{\log R_0}} \leqslant T \, e^{-\underline{C}\,\frac{T^2}{\log T}(1 + \mathcal{O}(\frac{\log\log T}{\log T}))} \,.$$

The other part of the sum is easily taken care of by using the bound

$$e^{-\underline{K}R^2 - \underline{C}\frac{T^2}{\log R}} \leqslant e^{-\underline{K}R^2} \,,$$

and estimating the corresponding sum. This finally proves that

$$\sum_{R \geqslant R_{\min}} e^{-\underline{K}R^2 - \underline{C}\frac{T^2}{\log R}} \leqslant e^{-\underline{C}\,\frac{T^2}{\log T}(1 - \mathcal{O}(\frac{\log\log T}{\log T}))} \,.$$

3.4. **Tail estimate: Lower bound.** The proof of the lower bound stated in Theorem 2.2 is very similar. From (8) and Lemma 5.2,

$$\langle \chi(\phi_i > T) \,\big|\, |\phi_j| \leqslant a, \, \forall j \in A \rangle_{\Lambda_N}^0 \geqslant \langle \chi(\phi_i > T + a) \rangle_{A^c}^0 \,.$$

To bound $\langle \chi(\phi_i > T + a) \rangle_{A^c}^0$, we'll use an approach similar to that of [10]. For any profile $\psi \in \mathbb{R}^{\mathbb{Z}^2}$ with $\psi_k = 0$ for all $k \notin A^c$, introducing the measure $\mu_{A^c}^\psi \triangleq \mu_{A^c}^0(\cdot + \psi)$, the following well-known inequality holds (see [6], Exercise 3.2.23, for example),

$$\log \frac{\mu_{A^c}^0(\phi_i > T + a)}{\mu_{A^c}^\psi(\phi_i > T + a)} \geqslant -\frac{H(\mu_{A^c}^\psi \,|\, \mu_{A^c}^0) + e^{-1}}{\mu_{A^c}^\psi(\phi_i > T + a)} \,,$$

where

$$H(\mu \,|\, \nu) = \begin{cases} \int \mathrm{d}\mu \log \frac{\mathrm{d}\mu}{\mathrm{d}\nu} & \text{if } \mu \ll \nu \\ \infty & \text{otherwise} \end{cases}$$

is the relative entropy of μ w.r.t. ν. Restricting our attention to profiles ψ satisfying $\psi_i \geqslant T + a$, we immediately get that

$$\mu_{A^c}^\psi(\phi_i > T + a) \geqslant \mu_{A^c}^0(\phi_i > 0) = \tfrac{1}{2}.$$

It therefore remains to estimate the relative entropy. We have

$$\begin{aligned} \mu_{A^c}^\psi(\log \frac{\mathrm{d}\mu_{A^c}^\psi}{\mu_{A^c}^0}) &= \mu_{A^c}^\psi\left(\sum_{\langle kl \rangle_r} \big(V(\phi_k - \phi_l) - V(\phi_k - \psi_k - \phi_l + \psi_l) \big) \right) \\ &= \mu_{A^c}^0\left(\sum_{\langle kl \rangle_r} \big(V(\phi_k - \phi_l + \psi_k - \psi_l) - V(\phi_k - \phi_l) \big) \right) \\ &\leqslant \sum_{\langle kl \rangle_r} \mu_{A^c}^0\big((\psi_k - \psi_l) V'(\phi_k - \phi_l) + \frac{\overline{c}}{2}(\psi_k - \psi_l)^2 \big) \\ &= \frac{\overline{c}}{2} \sum_{\langle kl \rangle_r} (\psi_k - \psi_l)^2 \,, \end{aligned}$$

where we used a Taylor expansion, the boundedness assumption and the symmetry of the measure $\mu_{A^c}^0$. Optimization over the profiles can be done using the following identity,

$$\inf\{\sum_{\langle kl \rangle_r}(\psi_k - \psi_l)^2 \,:\, \psi \in \mathbb{R}^{\mathbb{Z}^2},\, \psi \equiv 0 \text{ off } A^c,\, \psi_i \geqslant T + a\} = (T+a)^2 \operatorname{cap}_A^r(\{i\})$$

$$= (T+a)^2\, g_A^r(i,i)^{-1},$$

with $\operatorname{cap}_A^r(\{i\})$ the capacity of the set $\{i\}$ w.r.t. the (transient) RW on $\mathbb{Z}^2$ with jump-rates 1 between sites t, t' such that $\|t - t'\|_1 \leqslant r$ and killed upon entering A, and $g_A^r(i,i)$ the Green function of this RW (the last identity is standard, see e.g. **E**25.1 in [14]). Therefore, using $g_A^r(i,i) \leqslant C \log R$ for some $C > 0$, we finally get the existence of some constant $C' > 0$ such that

$$\mu_{A^c}^0(\phi_i > T + a) \geqslant \exp\{-\frac{\bar{c}}{C'}\frac{(T+a)^2}{\log R}\}. \tag{9}$$

To obtain the desired lower bound, it suffices then to restrict the sum over R in (8) to the single term $R = \overline{R}$ (which is possible by the hypothesis on i), and use (9).

3.5. **Unbounded boundary conditions.** We prove Proposition 2.2. Let us write $W_N(x) = W(x - f(N))$. Then $\operatorname{var}_{\Lambda_N}^{f(N),W}(\phi_i) = \operatorname{var}_{\Lambda_N}^{0,W_N}(\phi_i)$ and $\operatorname{cov}_{\Lambda_N}^{f(N),W}(\phi_i, \phi_j) = \operatorname{cov}_{\Lambda_N}^{0,W_N}(\phi_i, \phi_j)$. Clearly,

$$\mu_{\Lambda_N}^{0,W_N}(\exists k \in \Lambda_N,\, \phi_k > f(N)/2) \leqslant e^{b|\Lambda_N|}\, \mu_{\Lambda_N}^0(\exists k \in \Lambda_N,\, \phi_k > f(N)/2)$$

$$\leqslant |\Lambda_N| e^{b|\Lambda_N| - C\frac{f(N)^2}{\log N}}, \tag{10}$$

for some constant $C > 0$. Therefore, using (10), Cauchy-Schwarz and Jensen's inequality, we get, for large enough ζ,

$$\langle \phi_i^2 \rangle_{\Lambda_N}^{0,W_N} \geqslant \tfrac{1}{2}\langle \phi_i^2 \mid \phi_k \leqslant \tfrac{1}{2}f(N)\, \forall k \in \Lambda_N \rangle_{\Lambda_N}^0 \geqslant \tfrac{1}{4}\langle \phi_i^2 \rangle_{\Lambda_N}^0.$$

We also have

$$\langle \phi_i \rangle_{\Lambda_N}^{0,W_N} \leqslant \langle \phi_i \mid \phi_k \leqslant \tfrac{1}{2}f(N)\, \forall k \in \Lambda_N \rangle_{\Lambda_N}^0 + \langle \phi_i \mid \exists k \in \Lambda_N:\ \phi_k > \tfrac{1}{2}f(N) \rangle_{\Lambda_N}^0\, |\Lambda_N| e^{b|\Lambda_N| - C\frac{f(N)^2}{\log N}}$$

$$\leqslant \langle \phi_i \rangle_{\Lambda_N}^0 + \langle \phi_i \rangle_{\Lambda_N}^{f(N),W_N}\, |\Lambda_N| e^{b|\Lambda_N| - C\frac{f(N)^2}{\log N}}$$

$$\leqslant f(N)|\Lambda_N| e^{b|\Lambda_N| - C\frac{f(N)^2}{\log N}},$$

by FKG inequalities. Since the reverse Brascamp-Lieb inequality [5] implies that $\operatorname{var}_{\Lambda_N}^0(\phi_i) > \frac{C}{\bar{c}}\log N$ for some constant $C > 0$, this prove the claim about the variance. The proof for the covariance follows the same lines; notice that in the Gaussian case it is easy to get lower bounds on $\operatorname{cov}_{\Lambda_N}^{0,\star}(\phi_i, \phi_j)$ (see Proposition 1.6.7 in [13]).

4. Proof of the main estimate

This section is devoted to the proof of Proposition 4.1, which is the main estimate of this paper, and the starting point for the results of [12]. The most important is the first statement, but the other also appear to be useful. This proposition roughly states that an arbitrarily weak pinning potential is sufficient to decrease (strictly) the free energy; its power, however, lies in the fact that it is not restricted to well-behaved subsets (in the sense of Van Hove for example), but even applies to "one-dimensional" ones.

We say that a set $D \subseteq \mathbb{Z}^2$ is *M-connected*, if, for any $x, y \in D$, there exists an ordered sequence $(t_0 \equiv x, t_1, \ldots, t_n \equiv y)$ of sites of D such that $\|t_k - t_{k-1}\|_1 \leqslant M$, for all $k = 1, \ldots, n$. The *diameter* of a set D is defined by $\operatorname{diam} D = \max_{x,y \in D} \|x - y\|_1$.

Recall that ν is the probability measure on $\{A \subseteq \Lambda_N\}$ defined by the weights

$$\nu(A) = (e^b - 1)^{|A|} \mathbf{Z}_{\Lambda_N}(A) / \mathbf{Z}_{\Lambda_N}^W$$

in the square-well case and

$$\nu(A) = e^{J|A|} \mathbf{Z}_{A^c}^0 / \mathbf{Z}_{\Lambda_N}^J$$

in the δ-pinning case.

Proposition 4.1.
1. Consider both square-well and δ-pinning and set $2(e^b - 1)a = e^J$. Let $B \subseteq \Lambda_N$ be M-connected and such that $\operatorname{diam} B \geqslant \left(e^J \sqrt{c}\right)^{-C(M)}$ for some $C(M)$ large enough. Then, there exists $K = K(e^J \sqrt{c}, M)$, independent of B, such that

$$\nu(\{A : A \cap B = \varnothing\}) \leqslant \exp\{-K |B|\}.$$

Moreover, if $e^J \sqrt{c}$ is small enough, then there exists $C_6 = C_6(M)$ such that $K > (e^J \sqrt{c})^{C_6}$.
2. Let $B \subseteq \Lambda_N$. In the case of the square-well potential,

$$\nu(\{A : A \cap B = \varnothing\}) \geqslant \exp\{-b |B|\},$$

while in the case of δ-pinning,

$$\nu(\{A : A \cap B = \varnothing\}) \geqslant (1 + e^J)^{-|B|}.$$

Proof. Let us first prove 1. We introduce the following notations:

$$D^k \triangleq \{t \in \Lambda_N : \mathrm{d}_1(t, D) \leqslant k\},$$
$$\partial^{\mathrm{ext}} D \triangleq D^1 \setminus D,$$

and the two events: $\mathcal{E}_k = \{A : A \cap B^k = \varnothing\}$, $\mathcal{I}_k = \{A : (A \cup \partial^{\mathrm{ext}} \Lambda_N) \cap \partial^{\mathrm{ext}} B^k \neq \varnothing\}$. What we want to obtain is a upper bound on

$$\nu(\{A : A \cap B = \varnothing\}) = \sum_{k \geqslant 0} \nu(\mathcal{E}_k \cap \mathcal{I}_k) \leqslant \sum_{k \geqslant 0} \nu(\mathcal{E}_k \mid \mathcal{I}_k).$$

Observe that B^k is also M-connected and $\mathrm{diam}B^k > \mathrm{diam}B$. We can write

$$
\nu(\mathcal{E}_k \,|\, \mathcal{I}_k) = \frac{\displaystyle\sum_{A \in \mathcal{E}_k \cap \mathcal{I}_k} (e^b - 1)^{|A|} \mathbf{Z}_{\Lambda_N}(A)}{\displaystyle\sum_{A \in \mathcal{I}_k} (e^b - 1)^{|A|} \mathbf{Z}_{\Lambda_N}(A)}
$$

$$
= \frac{\displaystyle\sum_{A \in \mathcal{E}_k \cap \mathcal{I}_k} (e^b - 1)^{|A|} \mathbf{Z}_{\Lambda_N}(A)}{\displaystyle\sum_{C \subseteq B^k} (e^b - 1)^{|C|} \sum_{A \in \mathcal{E}_k \cap \mathcal{I}_k} (e^b - 1)^{|A|} \mathbf{Z}_{\Lambda_N}(A \cup C)}
$$

$$
= \left\{ \sum_{C \subseteq B^k} (e^b - 1)^{|C|} \sum_{A \in \mathcal{E}_k \cap \mathcal{I}_k} \rho(A) \frac{\mathbf{Z}_{\Lambda_N}(A \cup C)}{\mathbf{Z}_{\Lambda_N}(A)} \right\}^{-1}
$$

$$
\leqslant \left\{ \sum_{C \subseteq B^k} (e^b - 1)^{|C|} \inf_{A \in \mathcal{E}_k \cap \mathcal{I}_k} \frac{\mathbf{Z}_{\Lambda_N}(A \cup C)}{\mathbf{Z}_{\Lambda_N}(A)} \right\}^{-1}, \tag{11}
$$

where $\rho(A) \triangleq \nu(A \,|\, \mathcal{E}_k \cap \mathcal{I}_k)$.

One has therefore to bound the ratio of partition functions. If we enumerate the elements of C, say $C = \{t_1, \ldots, t_{|C|}\}$, and define $A_k \triangleq A \cup \{t_1, \ldots, t_k\}$, we get

$$
\frac{\mathbf{Z}_{\Lambda_N}(A \cup C)}{\mathbf{Z}_{\Lambda_N}(A)} = \frac{\mathbf{Z}_{\Lambda_N}(A_1)}{\mathbf{Z}_{\Lambda_N}(A_0)} \frac{\mathbf{Z}_{\Lambda_N}(A_2)}{\mathbf{Z}_{\Lambda_N}(A_1)} \cdots \frac{\mathbf{Z}_{\Lambda_N}(A_{|C|})}{\mathbf{Z}_{\Lambda_N}(A_{|C|-1})}.
$$

But, using Lemmas 5.4 and 5.5,

$$
\frac{\mathbf{Z}_{\Lambda_N}(A_k)}{\mathbf{Z}_{\Lambda_N}(A_{k-1})} = \mu^0_{\Lambda_N}(|\phi_{t_k}| \leqslant a \,|\, |\phi_j| \leqslant a, \,\forall j \in A_{k-1}) \geqslant \tfrac{1}{2}\mu^0_{A^c_{k-1}}(|\phi_{t_k}| \leqslant a) \geqslant \frac{a}{8\langle|\phi_{t_k}|\rangle^0_{A^c_{k-1}}}.
$$

Therefore,

$$
\frac{\mathbf{Z}_{\Lambda_N}(A \cup C)}{\mathbf{Z}_{\Lambda_N}(A)} \geqslant \prod_{k=1}^{|C|} \frac{a}{8\langle|\phi_{t_k}|\rangle^0_{A^c_{k-1}}}.
$$

To go further, we need to use the properties that B^k inherited from B. Let $l \in \mathbb{N}$ large enough (in particular $l \gg M$), but small compared to $\mathrm{diam}B$; we consider a grid of spacing l in Λ, with cells $\mathcal{C}_i$. Observe that there exists two numbers $\nu \in (0,1]$ and $\rho \in (0,1]$, independent of the set B^k and of l, such that the following properties hold: There exist two families of indices $\tilde{\mathbb{J}} \subseteq \mathbb{J}$ with $\{\mathcal{C}_j, j \in \mathbb{J}\}$ a connected set of cells, and $|\tilde{\mathbb{J}}| \geqslant \rho|\mathbb{J}|$, such that

- $B^k \subseteq \bigcup_{j \in \mathbb{J}} \mathcal{C}_j$,
- $B^k \cap \mathcal{C}_j \neq \varnothing$, for all $j \in \mathbb{J}$,
- $|B^k \cap \mathcal{C}_j| > \frac{\nu}{M}l$, for all $j \in \tilde{\mathbb{J}}$.

Indeed, the first statements are a simple consequence of the M-connectedness of B^k, and the last one is proven in the following way. Let $\{\mathcal{D}_i, i = 1, \ldots, N_D\}$ be a set of disjoint square boxes in $\mathbb{Z}^2$, built with exactly 9 cells of the grid defined above, and such that the middle-cell of each such box belongs to $\{\mathcal{C}_j, j \in \mathbb{J}\}$. We suppose that these boxes are chosen in such a way as to maximize N_D under these constraints. Then

- At most $16N_D$ cells of $\{\mathcal{C}_j, j \in \mathbb{J}\}$ are outside every $\mathcal{D}_i$.
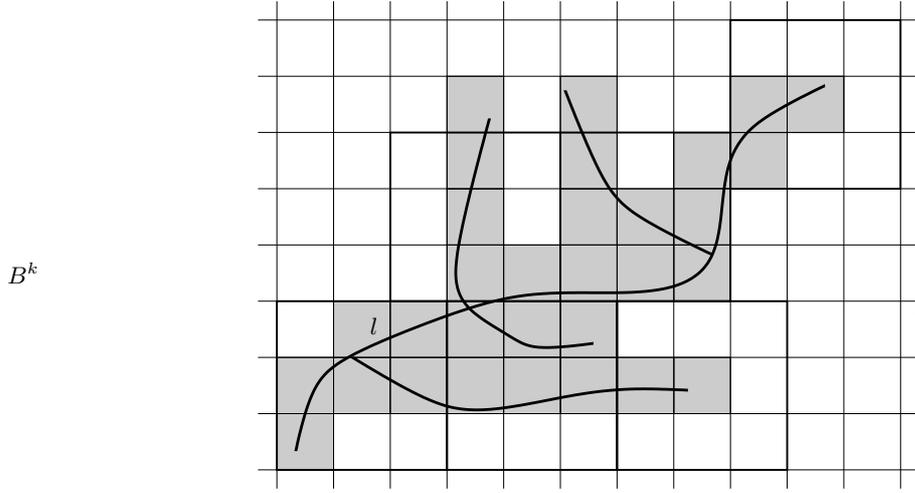
FIGURE 1. The shaded cells represents the set $\{\mathcal{C}_j,\, j \in \mathbb{J}\}$; the six large 3×3 squares represent the cells $\{\mathcal{D}_i,\, i = 1, \ldots, N_D\}$. The summation will be done on all sets $C \subseteq B^k$ containing exactly one site in each of the cells $\mathcal{C}_j,\, j \in \mathbb{J}$.

• Each $\mathcal{D}_i$ contains at most 9 cells of $\{\mathcal{C}_j,\, j \in \mathbb{J}\}$.

Therefore, $9N_D + 16N_D \geqslant |\mathbb{J}|$, i.e.

$$N_D \geqslant \tfrac{1}{25}|\mathbb{J}|\,.$$

Now, $|B^k \cap \mathcal{D}_i| > l/M$, for all $i = 1, \ldots, N_D$. Consequently, each box $\mathcal{D}_i$ contains at least one cell $\mathcal{C}$ with $|B^k \cap \mathcal{C}| \geqslant \frac{1}{9M}l$. Choosing $\nu = \frac{1}{9}$, this implies that

$$|\tilde{\mathbb{J}}| \geqslant \tfrac{1}{9}N_D \geqslant \tfrac{1}{225}|\mathbb{J}|\,,$$

so that we can take $\rho = \frac{1}{225}$.

We'll restrict the summation in (11) on sets $C \subseteq B^k$ which satisfy

$$|C \cap \mathcal{C}_i| = 1\,, \quad \forall i \in \mathbb{J}\,.$$

We number the elements of C as above, but in such a way as to ensure that

$$\{\mathcal{C} \,:\, \mathcal{C} \ni t_i\,, 1 \leqslant i \leqslant k\}$$

is connected for all $1 \leqslant k \leqslant |C| = |\mathbb{J}|$. Then $\mathrm{d}_1(t_k, A_{k-1}) \leqslant \sqrt{5}l$, for all $k > 1$. We further ask that $\mathrm{d}_1(t_1, A \cup \partial^{\mathrm{ext}}\Lambda_N) \leqslant \sqrt{5}l$, which is always possible. Using this, we obtain

$$\frac{\mathbf{Z}_{\Lambda_N}(A \cup C)}{\mathbf{Z}_{\Lambda_N}(A)} \geqslant \left(\frac{a\overline{K}\sqrt{c}}{\sqrt{\log l}}\right)^{|\mathbb{J}|}.$$

Indeed, (5) implies

$$\langle|\phi_{t_k}|\rangle^0_{A^{\mathrm{c}}_{k-1}} \leqslant \big[\langle\phi^2_{t_k}\rangle^0_{A^{\mathrm{c}}_{k-1}}\big]^{\frac{1}{2}} \leqslant \big[\tfrac{1}{c}\langle\phi^2_{t_k}\rangle^{0,\star}_{A^{\mathrm{c}}_{k-1}}\big]^{\frac{1}{2}} \leqslant \frac{\overline{K}}{\sqrt{c}}\sqrt{\log l}\,,$$

since, by construction, $\mathrm{d}_1(t_k, A_{t_{k-1}}) \leqslant \sqrt{5}l$, and the expectation value can be estimated using the random walk representation and standard results about irreducible, symmetric

random walk, see **P**11.6 and **P**12.3 in [14] for example. Therefore,

$$\sum_{C \subseteq B^k} (e^b - 1)^{|C|} \inf_{A \in \mathcal{E}_k \cap \mathcal{I}_k} \frac{\mathbf{Z}_{\Lambda_N}(A \cup C)}{\mathbf{Z}_{\Lambda_N}(A)} \geqslant \sum_{\substack{C \subseteq B^k \\ |C \cap \mathcal{C}_i| = 1, \forall i \in \mathbb{J}}} (e^b - 1)^{|\mathbb{J}|} \left(\frac{a \overline{K} \sqrt{c}}{\sqrt{\log l}} \right)^{|\mathbb{J}|}$$

$$\geqslant (\nu l)^{\rho |\mathbb{J}|} \left(\frac{(e^b - 1) a \sqrt{c} \overline{K}}{\sqrt{\log l}} \right)^{|\mathbb{J}|}$$

$$= \left(\frac{(e^b - 1) a \sqrt{c} \overline{K} (\nu l)^{\rho}}{\sqrt{\log l}} \right)^{|\mathbb{J}|}$$

$$\geqslant e^{|\mathbb{J}|} \geqslant e^{|B|/l^2},$$

which implies

$$\nu(\mathcal{E}_k \,|\, \mathcal{I}_k) \leqslant \exp\{-\tilde{K}_l \,|B^k|\},$$

for some $\tilde{K}_l > 0$ independent of B^k (provided $\mathrm{diam} B \gg l \geqslant l_0(a(e^b-1)\sqrt{c}, M)$). Therefore, we finally have

$$\nu(\{A : A \cap B = \varnothing\}) \leqslant \exp\{-K \,|B|\},$$

for some $K > 0$. The explicit bound on K follows by optimizing over l_0 above; this also explains the constraint on $\mathrm{diam} B$.

Let us prove 2 in the case of the square-well potential; the proof for δ-pinning is identical. Proceeding as in (11), we can write

$$\sum_{\substack{A \subseteq \Lambda_N \\ A \cap B = \varnothing}} (e^b - 1)^{|A|} \frac{\mathbf{Z}_{\Lambda_N}(A)}{\mathbf{Z}_{\Lambda_N}^W} \geqslant \left\{ \sum_{C \subseteq B} (e^b - 1)^{|C|} \sup_{A \subseteq \Lambda_N \setminus B} \frac{\mathbf{Z}_{\Lambda_N}(A \cup C)}{\mathbf{Z}_{\Lambda_N}(A)} \right\}^{-1}$$

$$\geqslant \left\{ \sum_{C \subseteq B} (e^b - 1)^{|C|} \right\}^{-1} = e^{-b|B|}.$$

$\square$

5. Appendix: Proofs of some technical estimates

In this section, we give the proofs of several technical statements used in the previous ones. Since FKG inequality is used several times, we recall that, as a consequence of Corollary 1.7 in [11], measures of the form

$$\frac{\mu_{\Lambda}^h(\cdot \prod_{i \in \Lambda} f_i(\phi_i))}{\mu_{\Lambda}^h(\prod_{i \in \Lambda} f_i(\phi_i))}$$

are FKG. If $A \subseteq \Lambda \subseteq \mathbb{Z}^2$, use the notation $A^c = \Lambda \setminus A$.

Lemma 5.1. *Let g be a positive, even function which is increasing on $\mathbb{R}^+$ and such that $g(0) = 0$ and let $a > 0$. Then, for any $\Lambda \Subset \mathbb{Z}^2$, any $A \subseteq \Lambda$ and any $j \in A^c$,*

$$\langle g(\phi_j) \,\big|\, |\phi_k| \leqslant a \;\forall k \in A \rangle_{\Lambda}^0 \leqslant \langle g(\phi_j + a) \,\big|\, \phi_j \geqslant -a \rangle_{A^c}^0.$$

Proof. We introduce $g_\nearrow(\phi_j) = g(\phi_j \vee 0)$. Using symmetry, FKG twice and translation invariance, we can write

$$
\begin{aligned}
\langle g(\phi_j) \,|\, |\phi_k| \leqslant a \;\forall k \in A\rangle_\Lambda^0 &= \langle g_\nearrow(\phi_j) \,|\, |\phi_k| \leqslant a \;\forall k \in A, \, \phi_j \geqslant 0\rangle_\Lambda^0 \\
&\leqslant \langle g_\nearrow(\phi_j) \,|\, \phi_k = a \;\forall k \in A, \, \phi_j \geqslant 0\rangle_\Lambda^0 \\
&\leqslant \langle g_\nearrow(\phi_j) \,|\, \phi_k = a \;\forall k \in A, \, \phi_j \geqslant 0\rangle_\Lambda^a \\
&= \langle g_\nearrow(\phi_j + a) \,|\, \phi_k = 0 \;\forall k \in A, \, \phi_j \geqslant -a\rangle_\Lambda^0 \\
&= \langle g(\phi_j + a) \,|\, \phi_j \geqslant -a\rangle_{A^c}^0 .
\end{aligned}
$$

Let us explain how the two inequalities are obtained. Let $\lambda > 0$. Since $\prod_{k \in A} \chi(\phi_k > a - \lambda)$ and $g_\nearrow$ are increasing, and the measure

$$
\chi(\phi_j \geqslant 0) \prod_{k \in A} \chi(|\phi_k| \leqslant a) \, \mathrm{d}\mu_\Lambda^0
$$

is FKG,

$$
\langle g_\nearrow(\phi_j) \,|\, |\phi_k| \leqslant a, \, \forall k \in A, \, \phi_j \geqslant 0\rangle_\Lambda^0 \leqslant \langle g_\nearrow(\phi_j) \,|\, |\phi_k| \in (a - \lambda, a], \, \forall k \in A, \, \phi_j \geqslant a\rangle_\Lambda^0 .
$$

Letting λ go to zero gives the first inequality. The second follows from the observation that $\mu_\Lambda^0(\mathrm{d}\phi) = \Phi_h(\phi)\mu_\Lambda^h(\mathrm{d}\phi)$, with $\Phi_h(\phi) = (\mathbf{Z}_\Lambda^h/\mathbf{Z}_\Lambda^0) \prod_{\substack{\langle ik\rangle_r \\ i \in \Lambda, \, k \notin \Lambda}} \exp\{V_{k-i}(\phi_i - h) - V_{k-i}(\phi_i)\}$

decreasing if $h > 0$. Indeed,

$$
\frac{\mathrm{d}}{\mathrm{d}\phi_i}(V_{k-i}(\phi_i - h) - V_{k-i}(\phi_i)) = -\int_{\phi_i - h}^{\phi_i} V_{k-i}''(x)\,\mathrm{d}x < 0 \,, \forall i \in \Lambda \,.
$$

$\square$

Lemma 5.2. *Let $T > 0$ and $a > 0$. Then, for all $\Lambda \Subset \mathbb{Z}^2$, $A \subseteq \Lambda$ and $i \in A^c$,*

$$
\mu_{A^c}^0(\phi_i > T + a) \leqslant \mu_\Lambda^0(\phi_i > T \,|\, |\phi_j| \leqslant a, \, \forall j \in A) \leqslant \mu_{A^c}^0(\phi_i > T - a) \,.
$$

Proof. The proof is completely similar to the previous one. We have

$$
\begin{aligned}
\langle \chi(\phi_i > T) \,|\, |\phi_j| \leqslant a \;\forall j \in A\rangle_{\Lambda_N}^0 &\leqslant \langle \chi(\phi_i > T) \,|\, |\phi_j| = a \;\forall j \in A\rangle_{\Lambda_N}^0 \\
&\leqslant \langle \chi(\phi_i > T) \,|\, |\phi_j| = a \;\forall j \in A\rangle_{\Lambda_N}^a \\
&\leqslant \langle \chi(\phi_i > T - a)\rangle_{A^c}^0 \,,
\end{aligned}
$$

and

$$
\begin{aligned}
\langle \chi(\phi_i > T) \,|\, |\phi_j| \leqslant a \;\forall j \in A\rangle_{\Lambda_N}^0 &\geqslant \langle \chi(\phi_i > T) \,|\, |\phi_j| = -a \;\forall j \in A\rangle_{\Lambda_N}^0 \\
&\geqslant \langle \chi(\phi_i > T) \,|\, |\phi_j| = -a \;\forall j \in A\rangle_{\Lambda_N}^{-a} \\
&\geqslant \langle \chi(\phi_i > T + a)\rangle_{A^c}^0 \,.
\end{aligned}
$$

$\square$

Lemma 5.3. *For any $\Lambda \Subset \mathbb{Z}^2$, $a > 0$ and $A \subseteq \Lambda$,*

$$
\langle \phi_i^2 \,|\, |\phi_j| \leqslant a, \, \forall j \in A\rangle_\Lambda^0 \leqslant 4a^2 + 4\langle \phi_i^2\rangle_{A^c}^0 \,.
$$

Proof. This follows easily from Lemma 5.1 with $g(x) = x^2$, and FKG inequality which yields

$$\langle ((\phi_i + a) \vee 0)^2 \,|\, \phi_i \geqslant -a \rangle^0_{A^c} \leqslant \langle ((\phi_i + a) \vee 0)^2 \,|\, \phi_i \geqslant 0 \rangle^0_{A^c}.$$

$\square$

Lemma 5.4. *For any $\Lambda \Subset \mathbb{Z}^2$, $a > 0$ and $A \subseteq \Lambda$,*

$$\mu^0_\Lambda(|\phi_i| \leqslant a \,|\, |\phi_j| \leqslant a, \, \forall j \in A)^0_\Lambda \geqslant \tfrac{1}{2}\mu^0_{A^c}(|\phi_i| \leqslant a).$$

Proof. Lemma 5.1 with $g(x) = \chi(|x| > a)$ implies that

$$
\begin{aligned}
\mu^0_\Lambda(|\phi_i| \leqslant a \,|\, |\phi_j| \leqslant a, \, \forall j \in A)^0_\Lambda &\geqslant \mu^0_{A^c}(|\phi_i + a| \leqslant a \,|\, \phi_i \geqslant -a) \\
&= \mu^0_{A^c}(\phi_i \leqslant 0 \,|\, \phi_i \geqslant -a) \\
&\geqslant \mu^0_{A^c}(-a \leqslant \phi_i \leqslant 0) \\
&= \tfrac{1}{2}\mu^0_{A^c}(|\phi_i| \leqslant a).
\end{aligned}
$$

$\square$

Lemma 5.5. *For any $\Lambda \Subset \mathbb{Z}^2$, $a > 0$ and $i \in \Lambda$,*

$$\mu^0_\Lambda(|\phi_i| \leqslant a) \geqslant \frac{a}{4\langle |\phi_i| \rangle^0_\Lambda} \wedge \tfrac{1}{2}.$$

Proof. The proof follows from the following elementary result, which is proved in [7]: *Let X be a random variable whose density under $\mathbb{P}$ is even and decreasing on $\mathbb{R}^+$. Then $\mathbb{P}[|X| \leqslant a] \geqslant \frac{a}{4\mathbb{E}[|X|]} \wedge \tfrac{1}{2}$, where $\mathbb{E}[\cdot]$ is the expectation value with respect to $\mathbb{P}$.*

Let F_j be the density of ϕ_j under μ^0_Λ. The evenness is obvious; let us check the monotonicity.

$$F_j(x) = \frac{1}{\mathbf{Z}^0_\Lambda} \int \prod_{t \in \mathbb{R}^{\Lambda \setminus \{j\}}} d\phi_t \prod_{\substack{\langle kl \rangle_r \\ k,l \in \Lambda \setminus \{j\}}} e^{-V_{l-k}(\phi_k - \phi_l)} \prod_{\substack{\langle jk \rangle_r \\ k \in \Lambda}} e^{-V_{k-j}(\phi_k - x)} \prod_{\substack{\langle jk \rangle_r \\ k \notin \Lambda}} e^{-V_{k-j}(x)} \prod_{\substack{\langle kl \rangle_r \\ k \in \Lambda \setminus \{j\} \\ l \notin \Lambda}} e^{-V_{l-k}(\phi_k)},$$

$$F'_j(x) = \tilde{C}_2 \sum_{\substack{\langle jk \rangle_r \\ k \notin \Lambda}} (-V'_{k-j}(x)) + \tilde{C}_3 \sum_{\substack{\langle jk \rangle_r \\ k \in \Lambda}} \langle V'_{k-j}(\phi_k - x) \,|\, \phi_j = x \rangle^0_\Lambda,$$

where $\tilde{C}_2$ and $\tilde{C}_3$ are two positive constants. Now, for $x \geqslant 0$, $V'_{k-j}(x) \geqslant 0$ and therefore the first term is negative. By FKG,

$$\langle V'_{k-j}(\phi_k - x) \,|\, \phi_j = x \rangle^0_\Lambda = \langle V'_{k-j}(\phi_k) \,|\, \phi_j = 0 \rangle^{-x}_\Lambda \leqslant \langle V'_{k-j}(\phi_k) \rangle^0_{\Lambda \setminus \{j\}} = 0,$$

since V'_{k-j} is increasing and odd. Consequently, $F'_j(x) \leqslant 0$ for $x \geqslant 0$. $\square$

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