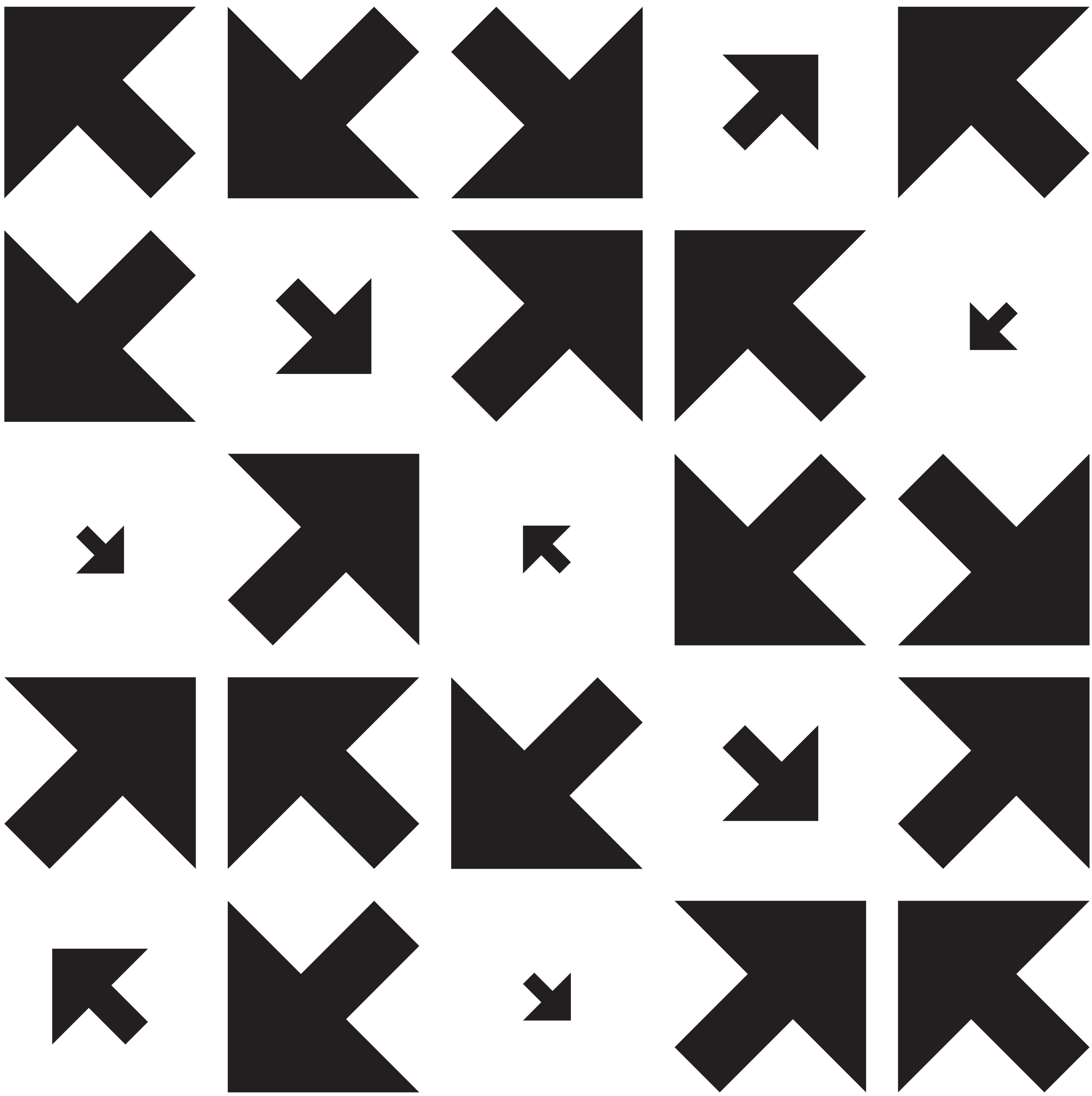
Statistical Mechanics of Lattice Systems

S. Friedli and Y. Velenik

A Concrete Mathematical Introduction



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For the sunshine and smiles: Janet, Jean-Pierre, Mimi and Kathryn.

Aos amigos e colegas do Departamento de Matemática.

À Agnese, Laure et Alexandre, ainsi qu'à mes parents.

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Preface

Equilibrium statistical mechanics is a field that has existed for more than a century. Its origins lie in the search for a microscopic justification of equilibrium thermodynamics, and it developed into a well-established branch of mathematics in the second half of the twentieth century. The ideas and methods that it introduced to treat systems with many components have now permeated many areas of science and engineering, and have had an important impact on several branches of mathematics.

There exist many good introductions to this theory designed for physics undergraduates. It might however come as a surprise that textbooks addressing it from a *mathematically rigorous* standpoint have remained rather scarce. A reader looking for an introduction to its more advanced mathematical aspects must often either consult highly specialized monographs or search through numerous research articles available in peer-reviewed journals. It might even appear as if the mastery of certain techniques has survived from one generation of researchers to the next only by means of oral communication, through the use of chalk and blackboard...

It seems a general opinion that pedagogical introductory mathematically rigorous textbooks simply do not exist. This book aims at starting to bridge this gap. Both authors graduated in physics before turning to mathematical physics. As such, we have witnessed this lack from the student's point of view, before experiencing it, a few years later, from the teacher's point of view. Above all, this text aims to provide the material we would have liked to have at our disposal when entering this field.

Although our hope is that it will also be of interest to students in theoretical physics, this is in fact a book on *mathematical physics*. There is no general consensus on what the latter term actually refers to. In rough terms, what it means for us is: the analysis of problems originating in physics, at the level of rigor associated to mathematics. This includes the introduction of concepts and the development of tools enabling such an analysis. It is unfortunate that mathematical physics is often held in rather low esteem by physicists, many of whom see it as useless nitpicking and as dealing mainly with problems that they consider to be already fully understood. There are however very good reasons for these investigations. First, such an approach allows a very clear separation between the assumptions (the basic principles of the underlying theory, as well as the particulars of the model analyzed)

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and the actual derivation: once the proper framework is set, the entire analysis is done without further assumptions or approximations. This is essential in order to ensure that the phenomenon that has been derived is indeed a consequence of the starting hypotheses and not an artifact of the approximations made along the way. Second, to provide a complete mathematical analysis requires us to understand the phenomenon of interest in a much deeper and detailed way. In particular, it forces one to provide precise definitions and statements. This is highly useful in clarifying issues that are sometimes puzzling for students and, occasionally, researchers.

Let us emphasize two central features of this work.

- The first has to do with content. Equilibrium statistical mechanics has become such a rich and diverse subject that it is impossible to cover more than a fraction of it in a single book. Since our driving motivation is to provide an easily accessible introduction in a form suitable for self-study, our first decision was to focus on some of the most important and relevant examples rather than to present the theory from a broad point of view. We hope that this will help the reader build the necessary intuition, in concrete situations, as well as provide background and motivation for the general theory. We also refrained from introducing abstractions for their own sake and have done our best to keep the technical level as low as possible.
- The second central feature of this book is related to our belief that the main value of the proof of a theorem is measured by the extent to which it enhances understanding of the phenomena under consideration. As a matter of fact, the concepts and methods introduced in the course of a proof are often at least as important as the claim of the theorem itself. The most useful proof, for a beginner, is thus not necessarily the shortest or the most elegant one. For these reasons, we have strived to provide, throughout the book, the arguments we personally consider the most enlightening in the most simple manner possible.

These two features have shaped the book from its very first versions. (They have also contributed, admittedly, to the lengthiness of some chapters.) Together with the numerous illustrations and exercises, we hope that they will help the beginner to become familiarized with some of the central concepts and methods that lie at the core of statistical mechanics.

As underlined by many authors, one of the main purposes of writing a book should be one's own pleasure. Indeed, leading this project to its conclusion was by and large a very enjoyable albeit long journey! But, beyond that, the positive feedback we have already received from students and from colleagues who have used early drafts in their lectures, indicates that it may yet reach its goal, which is to help beginners enter this beautiful field...

Acknowledgements. This book benefited both directly and indirectly from the help and support of many colleagues. First and foremost, we would like to thank Charles Pfister who, as a PhD advisor, introduced both authors to this field of research many years ago. We have also learned much of what we know from our various co-authors during the last two decades. In particular, YV would like to express his thanks to Dima Ioffe, for a long, fruitful and very enjoyable ongoing collaboration.

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Finally, writing this book would have been considerably less enjoyable without the following fantastic pieces of open source software: bash, GNU/Linux (openSUSE and Ubuntu flavors), GCC, GIMP, git, GNOME, gnuplot, Inkscape, KDE, Kile, LaTeX 2_ε, PGF/Tikz, POV-Ray, Processing, Python, Sketch (for LaTeX), TeXstudio, Vim and Xfig.

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Conventions

$a \stackrel{\text{def}}{=} b$	a is defined as being b
$\mathbb{R}^d$	d-dimensional Euclidean space
$\mathbb{Z}^d$	d-dimensional cubic lattice
$\mathbb{R}_{\geq 0}$	nonnegative real numbers
$\mathbb{R}_{>0}$	positive real numbers
$\mathbb{Z}_{\geq 0}$	nonnegative integers: $0, 1, 2, 3, \ldots$
$\mathbb{N}, \mathbb{Z}_{>0}$	positive integers: $1, 2, 3, \ldots$
i	$\sqrt{-1}$
$\mathfrak{Re}\, z, \mathfrak{Im}\, z$	real and imaginary parts of $z \in \mathbb{C}$
$a \wedge b$	minimum of a and b
$a \vee b$	maximum of a and b
$\log$	natural logarithm, that is, in base $e = 2.718...$
$A \subset B$	A is a (not necessarily proper) subset of B
$A \subsetneq B$	A is a proper subset of B
$A \triangle B$	symmetric difference
$\#A, \lvert A \rvert$	number of elements in the set A (if A is finite). At several places, also used to denote the Lebesgue measure.
$\delta_{m,n}$	Kronecker symbol: $\delta_{m,n} = 1$ if $m = n$, 0 otherwise
δ_x	Dirac measure at x: $\delta_x(A) = 1$ if $x \in A$, 0 otherwise
$\lfloor x \rfloor$	largest integer smaller or equal to x
$\lceil x \rceil$	smallest integer larger or equal to x

Asymptotic equivalence of functions will follow the standard conventions. For functions f, g, defined in the neighborhood of x_0 (possibly $x_0 = \infty$),

$f(x) \sim g(x)$	means $\lim_{x \to x_0} \frac{\log f(x)}{\log g(x)} = 1$,
$f(x) \simeq g(x)$	means $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1$,
$f(x) \approx g(x)$	means $0 < \liminf_{x \to x_0} \frac{f(x)}{g(x)} \leq \limsup_{x \to x_0} \frac{f(x)}{g(x)} < \infty$
$f(x) = O(g(x))$	means $\limsup_{x \to x_0} \left\lvert \frac{f(x)}{g(x)} \right\rvert < \infty$,
$f(x) = o(g(x))$	means $\lim_{x \to x_0} \left\lvert \frac{f(x)}{g(x)} \right\rvert = 0$.

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As usual, A^B is identified with the set of all maps $f : B \to A$. A sequence of elements $a_n \in E$ will usually be denoted as $(a_n)_{n\geq 1} \subset E$. At several places, we will set $0\log 0 \stackrel{\text{def}}{=} 0$. Sums or products over empty families are defined as follows:

$$\sum_{i\in\varnothing} a_i \stackrel{\text{def}}{=} 0 \qquad \prod_{i\in\varnothing} a_i \stackrel{\text{def}}{=} 1\,.$$

Several important notations involving several geometrical notions on $\mathbb{Z}^d$ will be defined at the end of the introduction and at the beginning of Chapter 3.

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