7 Pirogov–Sinai Theory

As we have already discussed several times in previous chapters, a central task of equilibrium statistical physics is to characterize all possible macroscopic behaviors of the system under consideration, given the values of the relevant thermodynamic parameters. This includes, in particular, the determination of the *phase diagram* of the model. This can be tackled in at least two ways, as was already seen in Chapter 3. In the first approach, one determines the set of all infinite-volume Gibbs measures as a function of the parameters of the model. In the second approach, one considers instead the associated pressure and studies its analytic properties as a function of its parameters; of particular interest is the determination of the set of values of the latter at which the pressure fails to be differentiable.

Our goal in the present chapter is to introduce the reader to the *Pirogov–Sinai theory*, in which these two approaches can be implemented, at sufficiently low temperatures (or in other perturbative regimes), for a rather general class of models. This theory is one of the few frameworks in which first-order phase transitions can be established and phase diagrams constructed, under general assumptions.

To make the most out of this chapter, the reader should preferably be familiar with the results derived for the Ising model in Chapter 3, as those provide useful intuition for the more complex problems addressed here. He should also be familiar with the cluster expansion technique exposed in Chapter 5, the latter being the basic tool we will use in our analysis. However, although it might help, a thorough understanding of the theory of Gibbs Measures, as exposed in Chapter 6, is not required.

Conventions. We know from Corollary 6.41 that one-dimensional models with finite-range interactions do not exhibit phase transitions and thus possess a trivial phase diagram at all temperatures. We will therefore always assume, throughout the chapter, that $d \geq 2$.

It will once more be convenient to adopt the physicists' convention and let the inverse temperature β appear as a multiplicative constant in the Boltzmann weights and in the pressures. To lighten the notations, we will usually omit to mention β and the external fields, especially for partition functions.

7.1 Introduction

Most of Chapter 3 was devoted to the study of the phase diagram of the Ising model as a function of the inverse temperature β and magnetic field h. In particular, it was shown there that, at low temperature, the features that distinguish the regimes $h < 0$, $h = 0$, $h > 0$ are closely related to the *ground states* of the Ising Hamiltonian, that is, the configurations with lowest energy. These are given by η^- if $h < 0$, η^+ if $h > 0$ and both η^+ and η^- if $h = 0$ (we remind the reader that η^+ and η^- are the constant configurations $\eta_i^\pm = \pm 1$ for all $i \in \mathbb{Z}^d$). In dimension $d \geq 2$, the main features of the behavior of the model at low temperature can then be summarized as follows:

- When $h < 0$, resp. $h > 0$, there is a unique infinite-volume Gibbs measure: $\mathscr{G}(\beta, h) = \{\mu_{\beta,h}\}$. Moreover, the pressure $h \mapsto \psi_\beta(h)$ is differentiable (in fact: analytic) on these regions.
- At $h = 0$, a first-order phase transition occurs, characterized by the non-differentiability of the pressure:
$$\frac{\partial \psi_\beta}{\partial h^-}\Big|_{h=0} \neq \frac{\partial \psi_\beta}{\partial h^+}\Big|_{h=0}.$$
When $h = 0$, the system becomes sensitive to the choice of boundary condition, in the sense that imposing $+$ or $-$ boundary condition yields two distinct Gibbs measures in the thermodynamic limit,
$$\mu^+_{\beta,0} \neq \mu^-_{\beta,0}.$$
As seen when implementing Peierls' argument, at low temperature the typical configurations under each of these measures are described by small local deviations away from the ground state corresponding to the chosen boundary condition. Later, we will refer to this phenomenon as the *stability* of the two ground states (or of the two $+$ and $-$ boundary conditions) at the transition point.

These features can thus be summarized by the following picture:

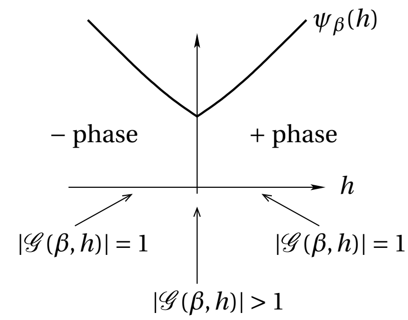

Figure 7.1: The phase diagram and pressure of the Ising model on $\mathbb{Z}^d$, $d \geq 2$, at low temperature.

We emphasize that the symmetry under the global spin flip enjoyed by the Ising model when $h = 0$ was a crucial simplifying feature when proving these results, and

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especially when implementing Peierls' argument (remember how spin flip symmetry was used on page 113).

In view of the above results, it is natural to wonder whether phase diagrams can be established rigorously for other models with more complicated interactions, in particular for models which do not enjoy any particular symmetry.

This is precisely the purpose of the Pirogov–Sinai theory (abreviated PST below). Even though the theory applies in more general frameworks, we will only discuss models with finite single-spin space and finite-range interactions. Let us just mention two examples, the second of which will be the main subject of this chapter. (Other fields of applications will be described in the bibliographical notes.)

7.1.1 A modified Ising model

Consider, for example, the following modification of the formal Hamiltonian of the Ising model:

$$-\sum_{\{i,j\}\in\mathscr{E}_{\mathbb{Z}^d}} \omega_i\omega_j + \epsilon \sum_{\{i,j,k\}} \omega_i\omega_j\omega_k - h\sum_{i\in\mathbb{Z}^d}\omega_i\,, \tag{7.1}$$

where the second sum is over all triples $\{i,j,k\}$ having diameter bounded by 1, and ϵ is a small, fixed parameter.

When $\epsilon = 0$, this model coincides with the Ising model. But, as soon as $\epsilon \neq 0$, the Hamiltonian is no longer invariant under a global spin flip when $h = 0$, and there is no reason anymore for $h = 0$ to be the point of coexistence. Nevertheless, when $|\epsilon|$ is small, η^- and η^+ are the only possible ground states (see Exercise 7.6), and one might expect this model and the Ising model to have similar phase diagrams, except that the former's might not be symmetric in h when $\epsilon \neq 0$.

The above modification of the Ising model can be studied rigorously using the methods of PST. It can be proved that, once β is sufficiently large, there exists for all ϵ (not too large) a unique transition point $h_t = h_t(\beta,\epsilon)$ such that the pressure $h \mapsto \psi^{\mathrm{modif}}_{\beta,\epsilon}(h)$ is differentiable when $h < h_t$ and when $h > h_t$, but is not differentiable at h_t:

$$\frac{\partial\psi^{\mathrm{modif}}_{\beta,\epsilon}}{\partial h^-}\Big|_{h=h_t} \neq \frac{\partial\psi^{\mathrm{modif}}_{\beta,\epsilon}}{\partial h^+}\Big|_{h=h_t}.$$

In fact, the theory also provides detailed information on the behavior of h_t as a function of β and ϵ and allows one to construct two distinct extremal Gibbs measures when $h = h_t$.

We will not discuss the properties of this model in detail here [1], but after having read the chapter, the reader should be able to provide rigorous proofs of the above claims.

7.1.2 Models with three or more phases

The PST is however not restricted to models with only two equilibrium phases. In this chapter, in order to remain as concrete as possible, a large part of the discussion will be done for one particular model of interest: the **Blume–Capel model** [2]. In the latter, spins take three values, $\omega_i \in \{+1, 0, -1\}$, and the formal Hamiltonian is defined by

$$\sum_{\{i,j\}\in\mathscr{E}_{\mathbb{Z}^d}} (\omega_i - \omega_j)^2 - h\sum_{i\in\mathbb{Z}^d}\omega_i - \lambda\sum_{i\in\mathbb{Z}^d}\omega_i^2\,. \tag{7.2}$$

Depending on the values of λ and h, this Hamiltonian has three possible ground states, given by the constant configurations η^+, η^0 and η^- (ignoring, for the moment, possible boundary effects). The set of pairs $(\lambda, h) \in \mathbb{R}^2$ then splits into three regions $\mathscr{U}^+, \mathscr{U}^0, \mathscr{U}^-$ such that $\eta^\#$ is the unique ground state when (λ, h) belongs to the interior of $\mathscr{U}^\#$. The picture represented on Figure 7.2a illustrates this, and is called the **zero-temperature phase diagram**.

We will prove that, at low temperature, the phase diagram is a small deformation of the latter (in a sense which will be made precise later); see Figure 7.2b.

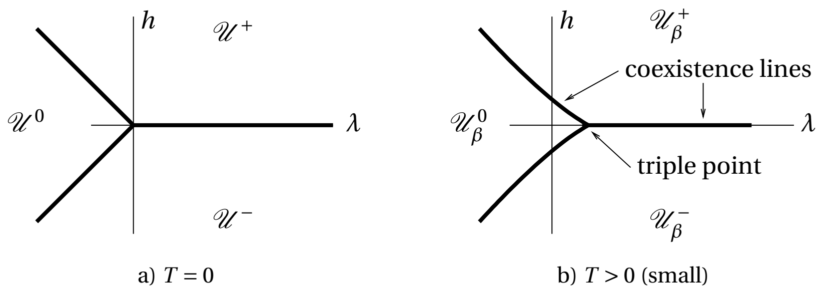

Figure 7.2: The Blume–Capel model at $T = 0$ ($\beta = \infty$) and small $T > 0$ ($\beta < \infty$, large). a) At zero temperature, the phase diagram is just a partition of the (λ, h) plane into regions with different ground state(s): when $(\lambda, h) \in \mathscr{U}^\#$, $\eta^\#$ is a ground state. On the boundaries of these regions, several ground states coexist. In particular, there are three ground states when $(\lambda, h) = (0,0)$. b) At low temperature, the phase diagram is a small and smooth deformation of the zero-temperature one. When $(\lambda, h) \in \mathscr{U}^\#_\beta$, an extremal Gibbs measure $\mu^\#_{\beta;\lambda,h}$ can be constructed using the boundary condition #; typical configurations under this measure are described by small deviations from the ground state $\eta^\#$. There exists a triple point $(\lambda_t, 0)$, at which these three distinct extremal Gibbs measures coexist. From the triple point emanate three coexistence lines. On each of the latter, exactly two of these measures coexist. . The rest of the diagram consists of uniqueness regions. The symmetry by a reflection across the λ-axis is due to the invariance of the Hamiltonian under the interchange of + and − spins. This phase diagram will be rigorously established in Section 7.4.

We will see in Theorem 7.36 that the pressure $(\lambda, h) \mapsto \psi_\beta(\lambda, h)$ is differentiable everywhere, except on the coexistence lines, across which its derivatives are discontinuous.

A qualitative plot of ψ_β can be found in Figure 7.3.

The analysis will also provide information on the structure of typical configurations, in Corollary 7.44.

Remark 7.1. The principles underlying the Pirogov–Sinai theory are rather general, robust and apply in many situations. Nevertheless, their current implementation requires perturbative techniques. As a consequence, this theory can provide precise information regarding the dependence of a model on its parameters only for regions of the parameters space which lie in a neighborhood of a regime that is already well understood. In this chapter, the latter will be the zero-temperature regime, and the results will thus only hold at sufficiently low temperatures (usually, *very* low temperatures). ◇

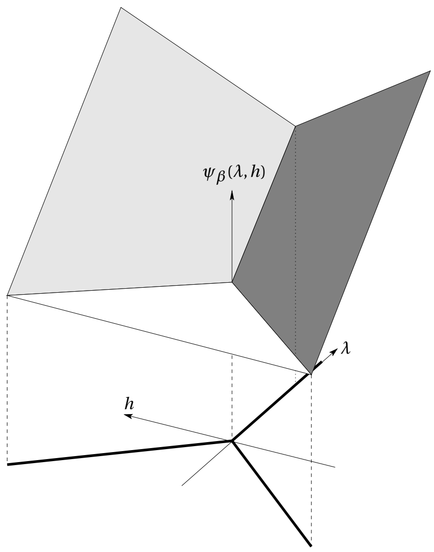

Figure 7.3: A qualitative plot of the pressure of the Blume–Capel at low temperature.

7.1.3 Overview of the chapter

We will first introduce the general notion of *ground state* in Section 7.2 and describe the basic structure that a model with finite-range interactions should have in order to enter the framework of the Pirogov–Sinai theory. Ultimately, this will lead to a representation of its partition function as a polymer model in Section 7.3.

In a second step, we will study those polymer models at low temperature and construct the phase diagram in Section 7.4. For the sake of concreteness, as in the rest of the book, we will avoid adopting too general a point of view and implement this construction only for the Blume–Capel model. The reason for this choice is that the latter is representative of the class of models to which this approach can be applied: Its analysis is sufficiently complicated to require the use of all the main ideas of PST, but simple enough to keep the discussion (and the notations) as elementary as possible. On the one hand, the absence of symmetry between the 0 and ± 1 spins makes it impossible to implement a “naive” Peierls' argument, as was done for the Ising model (remember how the ratio of partition functions was bounded on page 113). On the other hand, since this model includes two external fields (h and λ), its phase diagram has already a nontrivial structure, containing coexistence lines and a triple point, as shown on Figure 7.2.

We are confident that, once he has read carefully the construction of the phase diagram of the Blume–Capel model, the reader should be able to adapt the ideas to new situations.

7.1.4 Models with finite-range translation invariant interactions

The models to which PST applies are essentially those introduced in Section 6.3.2, with a potential Φ satisfying a set of extra conditions that will be described in the next section.

The distance on $\mathbb{Z}^d$ used throughout this chapter is the one associated to the norm $\|i\|_\infty \stackrel{\text{def}}{=} \max_{1\le k\le d} |i_k|$. The **diameter** of a set $B \subset \mathbb{Z}^d$, in particular, is defined by

$$\mathrm{diam}(B) \stackrel{\text{def}}{=} \sup\{d_\infty(i,j) : i,j \in B\},$$

where $d_\infty(i,j) \stackrel{\text{def}}{=} \|j-i\|_\infty$. We will use two notions of **boundary**: for $A \subset \mathbb{Z}^d$,

$$\partial^{\text{ex}} A \stackrel{\text{def}}{=} \{i \in A^{\text{c}} : d_\infty(i,A) \le 1\}, \tag{7.3}$$

$$\partial^{\text{in}} A \stackrel{\text{def}}{=} \{i \in A : d_\infty(i,A^{\text{c}}) \le 1\}. \tag{7.4}$$

As in Section 6.6, the **translation** by $i \in \mathbb{Z}^d$ will be denoted by θ_i and can act on configurations, events and measures.

We assume throughout that the single-spin space Ω_0 is *finite* and set, as usual, $\Omega_\Lambda \stackrel{\text{def}}{=} \Omega_0^\Lambda$, and $\Omega \stackrel{\text{def}}{=} \Omega_0^{\mathbb{Z}^d}$.

All the potentials $\Phi = \{\Phi_B\}_{B \Subset \mathbb{Z}^d}$ considered in this chapter will be of **finite range**,

$$r(\Phi) = \inf\{R > 0 : \Phi_B \equiv 0 \text{ for all } B \text{ with } \mathrm{diam}(B) > R\} < \infty,$$

and **invariant under translations**, meaning that

$$\Phi_{\theta_i B}(\theta_i \omega) = \Phi_B(\omega), \qquad \forall i \in \mathbb{Z}^d, \forall \omega \in \Omega.$$

The notations concerning Gibbs distributions associated to a potential Φ are those used in Section 6.3.2. For instance, the **Hamiltonian** in a region $\Lambda \Subset \mathbb{Z}^d$ is defined as usual by

$$\mathscr{H}_{\Lambda;\Phi}(\omega) \stackrel{\text{def}}{=} \sum_{\substack{B \Subset \mathbb{Z}^d: \\ B\cap\Lambda\neq\varnothing}} \Phi_B(\omega), \qquad \omega \in \Omega. \tag{7.5}$$

The partition function (denoted previously by $\mathbf{Z}^\eta_{\Lambda;\Phi}$, see (6.31)) will be denoted slightly differently, in order to emphasize its dependence on the set Λ, for reasons that will become clear later:

$$\mathbf{Z}^\eta_\Phi(\Lambda) \stackrel{\text{def}}{=} \sum_{\omega \in \Omega^\eta_\Lambda} \exp\bigl(-\beta \mathscr{H}_{\Lambda;\Phi}(\omega)\bigr). \tag{7.6}$$

We remind the reader that $\Omega^\eta_\Lambda \stackrel{\text{def}}{=} \{\omega \in \Omega : \omega_{\Lambda^{\text{c}}} = \eta_{\Lambda^{\text{c}}}\}$.

The **pressure** is obtained by considering the thermodynamic limit along a sequence $\Lambda \Uparrow \mathbb{Z}^d$:

$$\psi(\Phi) \stackrel{\text{def}}{=} \lim_{\Lambda \Uparrow \mathbb{Z}^d} \frac{1}{\beta|\Lambda|} \log \mathbf{Z}^\eta_\Phi(\Lambda). \tag{7.7}$$

In Theorem 6.79, we showed the existence of this limit along the sequence of boxes $\mathrm{B}(n)$, $n \to \infty$, for absolutely summable potentials. When the range is finite, existence can also be obtained by a simpler method.

Exercise 7.1. *Adapting the proof of Theorem 3.6, show that $\psi(\Phi)$ exists, depends neither on η nor on the sequence $\Lambda \Uparrow \mathbb{Z}^d$ and is convex.*

7.2 Ground states and Peierls' Condition

Loosely speaking, the main outcome of the Pirogov–Sinai theory is the determination of sufficient conditions that guarantee that typical configurations at (sufficiently low) positive temperatures are perturbations of those at zero temperature. As we already saw in the discussion of Section 1.4.3, typical configurations at zero temperature are those of minimal energy, that is, the ground states. Our first task is to find a suitable extension of this notion to infinite systems.

Remark 7.2. Note that what we call *ground states*, below, are in fact configurations (elements of Ω). This use of the word *state* should thus not be confused with that of earlier chapters, in which a state was a suitable linear functional acting on local functions. ⋄

Since $\mathscr{H}_{\Lambda;\Phi}$ is usually not defined when $\Lambda = \mathbb{Z}^d$, defining ground states as the configurations minimizing the total energy (on $\mathbb{Z}^d$) raises the same difficulty we already encountered in Section 6.1. The resolution of this problem is based on the same observation we made there: the *difference* of energy between two configurations coinciding everywhere outside a finite set is always well defined. This leads to characterizing a ground state as a configuration whose energy cannot be lowered by changing its value at finitely many vertices. To make this idea precise, we start by introducing the following notion: two configurations $\omega, \tilde\omega \in \Omega$ are **equal at infinity** if they differ only at finitely many points, that is, if there exists a finite region $\Lambda \Subset \mathbb{Z}^d$ such that

$$\tilde\omega_{\Lambda^c} = \omega_{\Lambda^c} .$$

(As in Chapter 6, we use ω_{Λ^c} to denote the restriction of ω to Λ^c.) When $\tilde\omega$ and ω are equal at infinity, we write $\tilde\omega \overset{\infty}{=} \omega$; in such a case, $\tilde\omega$ can be considered as a *local perturbation of* ω (and vice versa). Then, the **relative Hamiltonian** is defined by

$$\mathscr{H}_\Phi(\tilde\omega\,|\,\omega) \overset{\text{def}}{=} \sum_{B \Subset \mathbb{Z}^d} \bigl\{\Phi_B(\tilde\omega) - \Phi_B(\omega)\bigr\} .$$

When $\tilde\omega \overset{\infty}{=} \omega$, the sum on the right-hand side is well defined, since it contains only finitely many non-zero terms (remember that $r(\Phi) < \infty$).

Definition 7.3. *$\eta \in \Omega$ is called a* ***ground state*** *(for Φ) if*

$$\mathscr{H}_\Phi(\omega\,|\,\eta) \geq 0 \qquad \textit{for each} \quad \omega \overset{\infty}{=} \eta .$$

We denote the set of ground states for Φ by $g(\Phi)$.

Note that physically equivalent potentials (see Remark 6.17) yield the same relative Hamiltonian, and thus define the same set of ground states.

We will be mostly interested in *periodic* ground states. A configuration $\omega \in \Omega$ is **periodic** if there exist positive integers $l_1, \dots, l_d$ such that $\theta_{l_k \mathbf{e}_k}\omega = \omega$ for each $k = 1, \dots, d$ (remember that $\{\mathbf{e}_1, \dots, \mathbf{e}_d\}$ is the canonical basis of $\mathbb{R}^d$). The unique d-tuple $(l_1, \dots, l_d)$, in which each l_k is the smallest integer for which that property is satisfied, is called the **period** of ω. The set of periodic configurations is denoted by $\Omega^{\text{per}} \subset \Omega$ and the set of **periodic ground states** for Φ by $g^{\text{per}}(\Phi) \overset{\text{def}}{=} g(\Phi) \cap \Omega^{\text{per}}$.

We now provide a more global characterization of ground states. For $\omega \in \Omega^{\text{per}}$, the limit

$$\mathsf{e}_\Phi(\omega) \overset{\text{def}}{=} \lim_{n\to\infty} \frac{1}{|\mathsf{B}(n)|} \mathscr{H}_{\mathsf{B}(n);\Phi}(\omega)$$

clearly exists; it is called the **energy density of** ω.

Lemma 7.4. *Let* $\eta \in \Omega^{\mathrm{per}}$. *Then* $\eta \in g^{\mathrm{per}}(\Phi)$ *if and only if its energy density is minimal:*

$$\mathsf{e}_\Phi(\eta) = \underline{\mathsf{e}}_\Phi \stackrel{\text{def}}{=} \inf_{\omega\in\Omega^{\mathrm{per}}} \mathsf{e}_\Phi(\omega)\,.$$

Proof. Let us introduce $\widetilde{g}^{\mathrm{per}}(\Phi) \stackrel{\text{def}}{=} \{\omega \in \Omega^{\mathrm{per}} : \mathsf{e}_\Phi(\omega) = \underline{\mathsf{e}}_\Phi\}$.

We first assume that $\eta \in g^{\mathrm{per}}(\Phi)$. For all $\omega \in \Omega^{\mathrm{per}}$, we write

$$\mathsf{e}_\Phi(\omega) = \lim_{n\to\infty} \frac{1}{|\mathsf{B}(n)|}\big\{\mathscr{H}_{\mathsf{B}(n);\Phi}(\omega) - \mathscr{H}_{\mathsf{B}(n);\Phi}(\eta)\big\} + \mathsf{e}_\Phi(\eta)\,.$$

For all large n, define $\omega^{(n)} \stackrel{\text{def}}{=} \omega_{\mathsf{B}(n)}\eta_{\mathsf{B}(n)^c}$. Then $\omega^{(n)} \stackrel{\infty}{=} \eta$ and, since Φ has finite range,

$$\mathscr{H}_{\mathsf{B}(n);\Phi}(\omega) - \mathscr{H}_{\mathsf{B}(n);\Phi}(\eta) = \mathscr{H}_\Phi(\omega^{(n)}\,|\,\eta) + O(|\partial^{\mathrm{ex}}\mathsf{B}(n)|)\,. \tag{7.8}$$

Since $\mathscr{H}_\Phi(\omega^{(n)}\,|\,\eta) \geq 0$ and $\lim_{n\to\infty}\frac{|\partial^{\mathrm{ex}}\mathsf{B}(n)|}{|\mathsf{B}(n)|} = 0$, this proves that $\mathsf{e}_\Phi(\omega) \geq \mathsf{e}_\Phi(\eta)$. We conclude that $\eta \in \widetilde{g}^{\mathrm{per}}(\Phi)$.

Let us now assume that $\eta \in \widetilde{g}^{\mathrm{per}}(\Phi)$ and let ω be such that $\omega \stackrel{\infty}{=} \eta$. Since Φ has finite range, we can find k such that all the sets B that yield a non-zero contribution to $\mathscr{H}_\Phi(\omega\,|\,\eta)$ satisfy $B \subset \mathsf{B}(k)$, and such that $\omega_{\mathsf{B}(k)^c} = \eta_{\mathsf{B}(k)^c}$. Let ω^{per} be the periodic configuration obtained by tiling $\mathbb{Z}^d$ with copies of $\omega_{\mathsf{B}(k)}$ on all adjacent translates of $\mathsf{B}(k)$. Proceeding as above, we write

$$\begin{aligned}\mathsf{e}_\Phi(\omega^{\mathrm{per}}) &= \lim_{n\to\infty} \frac{1}{|\mathsf{B}(n)|}\big\{\mathscr{H}_{\mathsf{B}(n);\Phi}(\omega^{\mathrm{per}}) - \mathscr{H}_{\mathsf{B}(n);\Phi}(\eta)\big\} + \mathsf{e}_\Phi(\eta)\\ &= \frac{1}{|\mathsf{B}(k)|}\big\{\mathscr{H}_{\mathsf{B}(k);\Phi}(\omega^{\mathrm{per}}) - \mathscr{H}_{\mathsf{B}(k);\Phi}(\eta)\big\} + \mathsf{e}_\Phi(\eta)\\ &= \frac{1}{|\mathsf{B}(k)|}\mathscr{H}_\Phi(\omega\,|\,\eta) + \mathsf{e}_\Phi(\eta)\,.\end{aligned}$$

Since $\mathsf{e}_\Phi(\eta) \leq \mathsf{e}_\Phi(\omega^{\mathrm{per}})$, it follows that $\mathscr{H}_\Phi(\omega\,|\,\eta) \geq 0$. We conclude that $\eta \in g^{\mathrm{per}}(\Phi)$. □

Let us apply the above criterion to some examples. For reasons that will become clear later, we will temporarily denote the potential by Φ^0 rather than Φ.

Example 7.5. Let us consider the **nearest-neighbor Ising model in the absence of magnetic field**. Remember that, in this case, $\Omega_0 = \{\pm 1\}$ and, for all $B \Subset \mathbb{Z}^d$,

$$\Phi^0{}_B(\omega) \stackrel{\text{def}}{=} \begin{cases} -\omega_i\omega_j & \text{if } B = \{i,j\},\ i \sim j\,,\\ 0 & \text{otherwise.}\end{cases} \tag{7.9}$$

(We remind the reader that, in this chapter, the inverse temperature is kept outside the Hamiltonian.) Consider the constant (and thus periodic) configurations η^+ and η^-. Then, for all $\omega \stackrel{\infty}{=} \eta^\pm$,

$$\mathscr{H}_{\Phi^0}(\omega\,|\,\eta^\pm) = \sum_{\{i,j\}\in\mathscr{E}_{\mathbb{Z}^d}} (1 - \omega_i\omega_j) \geq 0\,, \tag{7.10}$$

which shows that $\eta^+, \eta^- \in g^{\mathrm{per}}(\Phi^0)$. The associated energy densities can be easily computed explicitly: $\mathsf{e}_{\Phi^0}(\eta^+) = \mathsf{e}_{\Phi^0}(\eta^-) = -d$. Moreover, any periodic configuration $\omega \neq \eta^\pm$ satisfies $\mathsf{e}_{\Phi^0}(\omega) > \mathsf{e}_{\Phi^0}(\eta^\pm)$. Therefore, $\eta^\pm$ are the only periodic ground states:

$$g^{\mathrm{per}}(\Phi^0) = \{\eta^+, \eta^-\}\,.$$

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There are, however, infinitely many other (nonperiodic) ground states (see Exercise 7.2). ⋄

Exercise 7.2. *Consider the Ising model on $\mathbb{Z}^2$ (still with no magnetic field). Fix $\mathbf{n} \in \mathbb{R}^2$ and define $\eta \in \Omega$ by $\eta_i = 1$ if and only if $\mathbf{n} \cdot i \geq 0$. Show that η and all its translates are ground states for the potential Φ^0 defined in* (7.9).

Example 7.6. Let us now consider the **Blume–Capel model in the absence of external fields**. Remember that, in this model, $\Omega_0 = \{-1, 0, +1\}$ and

$$\Phi^0{}_B(\omega) \stackrel{\text{def}}{=} \begin{cases} (\omega_i - \omega_j)^2 & \text{if } B = \{i,j\},\ i \sim j\,, \\ 0 & \text{otherwise.} \end{cases}$$

Let us consider again the constant configurations $\eta^+ \equiv +1$, $\eta^0 \equiv 0$ and $\eta^- \equiv -1$. Let $\# \in \{+,-,0\}$. Then, for all $\omega \stackrel{\infty}{=} \eta^\#$,

$$\mathscr{H}_{\Phi^0}(\omega \,|\, \eta^\#) = \sum_{\{i,j\} \in \mathscr{E}_{\mathbb{Z}^d}} (\omega_i - \omega_j)^2 \geq 0\,,$$

so that each $\eta^\#$ is a ground state. Since $\mathsf{e}_{\Phi^0}(\eta^+) = \mathsf{e}_{\Phi^0}(\eta^-) = \mathsf{e}_{\Phi^0}(\eta^0) = 0$ and any periodic, non-constant configuration ω has $\mathsf{e}_{\Phi^0}(\omega) > 0$, we conclude that $g^{\text{per}}(\Phi^0) = \{\eta^+, \eta^0, \eta^-\}$. ⋄

Additional examples will be discussed in Section 7.2.2.

Exercise 7.3. *Show that a model with a finite single-spin space and a finite-range potential always has at least one ground state.*

7.2.1 Boundaries of a configuration

From now on, we assume that the model under consideration has a *finite* number of periodic ground states:

$$g^{\text{per}}(\Phi) = \{\eta^1, \dots, \eta^m\}.$$

We use the symbol $\# \in \{1, 2, \dots, m\}$ to denote an arbitrary index associated to the ground states of the model. Since our goal is to establish the existence of phase transitions, we assume that Φ has at least two periodic ground states: $m \geq 2$.

In view of what was proved for the Ising model, one might expect a typical configuration of an infinite system (with potential Φ) at low temperature to consist of large regions on each of which the configuration coincides with some ground state $\eta^\# \in g^{\text{per}}(\Phi)$. Fix an integer $r > r(\Phi)$.

Definition 7.7. *Let $\omega \in \Omega$. A vertex $i \in \mathbb{Z}^d$ is* ***#-correct (in ω)*** *if*

$$\omega_j = \eta^\#_j, \qquad \forall j \in i + \mathsf{B}(r)\,.$$

The ***boundary of*** *ω is defined by*

$$\mathscr{B}(\omega) \stackrel{\text{def}}{=} \bigl\{ i \in \mathbb{Z}^d : i \text{ is not \#-correct in } \omega \text{ for any } \# \in \{1, \dots, m\} \bigr\}.$$

Before pursuing, let us make a specific choice for r. Let $(l^\#_1, \dots, l^\#_d)$ denote the period of $\eta^\#$. Until the end of this section, we use $r = r_*$, where

$$r_* \stackrel{\text{def}}{=} \text{least common multiple of } \bigl\{ l^\#_k : 1 \leq k \leq d,\ 1 \leq \# \leq m \bigr\} \text{ larger or equal to } r(\Phi)\,.$$

This choice implies that $\mathscr{B}(\omega)\cup\bigcup_{\#}\{\#\text{-correct vertices}\}$ forms a partition of $\mathbb{Z}^d$:

Lemma 7.8. *A vertex can be #-correct for at most one index #, and regions of #-correct and #′-correct vertices, #′ ≠ #, are separated by $\mathscr{B}(\omega)$, in the sense that, if i is #-correct and i' is #′-correct and if $i_1 = i, i_2, \dots, i_{n-1}, i_n = i'$ is a path such that $d_\infty(i_k, i_{k+1}) \le 1$, then there exists some $1 < k < n$ such that $i_k \in \mathscr{B}(\omega)$.*

Proof. Observe first that our choice of r_* implies that any cube of sidelength r_* contains at least one period of each ground state. The first claim follows immediately, since $i + \mathsf{B}(r_*)$ contains such a cube. For the second claim, note that if i_k, i_{k+1} are two vertices at distance 1, then $\{i_k + \mathsf{B}(r_*)\} \cap \{i_{k+1} + \mathsf{B}(r_*)\}$ also contains a cube of sidelength r_* and thus i_k and i_{k+1} can be #-correct only for the same label #. □

Since the boundary of a configuration contains all vertices at which the energy is higher than in the ground states, it is natural to try to bound the relative Hamiltonian with respect to a ground state in terms of the size of the boundary.

Lemma 7.9. *Let $\eta \in g^{\mathrm{per}}(\Phi)$. Then there exists a constant $C > 0$ (depending on Φ) such that, for any configuration ω such that $\omega \stackrel{\infty}{=} \eta$,*

$$\mathscr{H}_\Phi(\omega\,|\,\eta) \le C|\mathscr{B}(\omega)|\,. \tag{7.11}$$

Observe that

$$\mathscr{H}_{\Lambda;\Phi} = \sum_{\substack{B \Subset \mathbb{Z}^d:\\ B\cap\Lambda\neq\varnothing}} \Phi_B = \sum_{i\in\Lambda} \sum_{\substack{B \Subset \mathbb{Z}^d:\\ B\ni i}} \frac{1}{|B\cap\Lambda|}\Phi_B\,.$$

Introducing the functions

$$u_{i;\Phi} \stackrel{\text{def}}{=} \sum_{\substack{B \Subset \mathbb{Z}^d:\\ B\ni i}} \frac{1}{|B|}\Phi_B\,, \qquad i\in\mathbb{Z}^d\,,$$

we have

$$\Bigl|\mathscr{H}_{\Lambda;\Phi} - \sum_{i\in\Lambda} u_{i;\Phi}\Bigr| \le c|\partial^{\mathrm{ex}}\Lambda|\,, \tag{7.12}$$

for some constant c that depends on Φ. We also have

$$\|u_{i;\Phi}\|_\infty \le \|\Phi\| \stackrel{\text{def}}{=} \sum_{\substack{B \Subset \mathbb{Z}^d:\\ B\ni 0}} \frac{1}{|B|}\|\Phi_B\|_\infty\,.$$

In the proof of the above lemma, but also in other arguments, it will be convenient to use the partition $\mathscr{P}$ of $\mathbb{Z}^d$ into adjacent cubic boxes of linear size r_*, of the form $b_k = kr_* + \{0,1,2,\dots,r_*-1\}^d$, where $k\in\mathbb{Z}^d$. Since each of these boxes contains an integer number of periods of each $\eta^\# \in g^{\mathrm{per}}(\Phi)$, one has in particular, for all $k\in\mathbb{Z}^d$,

$$\frac{1}{|b_k|}\sum_{i\in b_k} u_{i;\Phi}(\eta^\#) = \mathsf{e}_\Phi(\eta^\#) = \underline{\mathsf{e}}_\Phi\,. \tag{7.13}$$

Proof of Lemma 7.9: Let $\omega \stackrel{\infty}{=} \eta$, and let $[\mathscr{B}](\omega)$ be the set of boxes $b \in \mathscr{P}$ whose intersection with $\mathscr{B}(\omega)$ is non-empty. Boxes b which are not part of $[\mathscr{B}](\omega)$ contain only correct vertices, and these are all correct for the same index # by Lemma 7.8. Then,

$$\begin{aligned}\mathscr{H}_\Phi(\omega\,|\,\eta) &= \sum_{i\in\mathbb{Z}^d}\{u_{i;\Phi}(\omega)-u_{i;\Phi}(\eta)\}\\ &= \sum_{b\in[\mathscr{B}](\omega)}\sum_{i\in b}\{u_{i;\Phi}(\omega)-u_{i;\Phi}(\eta)\}+\sum_{b\notin[\mathscr{B}](\omega)}\sum_{i\in b}\{u_{i;\Phi}(\omega)-u_{i;\Phi}(\eta)\}\,.\end{aligned}$$

The first double sum is upper-bounded by $2\|\Phi\| r_*^d|\mathscr{B}(\omega)|$. The second vanishes, since to each $b\notin[\mathscr{B}](\omega)$ corresponds some # such that $\omega_b=\eta_b^\#$, giving

$$\sum_{i\in b}\{u_{i;\Phi}(\omega)-u_{i;\Phi}(\eta)\}=|b|(\mathsf{e}_\Phi(\eta^\#)-\mathsf{e}_\Phi(\eta))=0\,. \qquad \square$$

For physical reasons, it is natural to expect that the energy of a configuration is proportional to the size of the boundary that separates regions with different periodic ground states, as happens in the Ising model. It is therefore natural to require that $\mathscr{H}_\Phi(\omega\,|\,\eta)$ should also grow proportionally to $|\mathscr{B}(\omega)|$. The notion that we will actually need is that of **thickened boundary**, defined by

$$\Gamma(\omega)\stackrel{\text{def}}{=}\bigcup\big\{i+\mathsf{B}(r_*)\,:\,i\in\mathscr{B}(\omega)\big\}\,. \tag{7.14}$$

The upper bound (7.11) implies of course that $\mathscr{H}_\Phi(\omega\,|\,\eta)\le C|\Gamma(\omega)|$ when $\omega\stackrel{\infty}{=}\eta$, but a corresponding *lower* bound does not hold in general. This turns out to be the main assumption of the Pirogov–Sinai theory:

Definition 7.10. Φ *is said to* ***satisfy Peierls' condition*** *if*

1. $g^{\mathrm{per}}(\Phi)$ *is finite, and*
2. *there exists a constant* $\rho>0$ *such that, for each* $\eta\in g^{\mathrm{per}}(\Phi)$*,*

$$\mathscr{H}_\Phi(\omega\,|\,\eta)\ge\rho|\Gamma(\omega)|\,,\qquad \textit{for all } \omega\stackrel{\infty}{=}\eta\,.$$

We call ρ ***Peierls' constant****.*

Peierls' condition can be violated even in simple models. An example will be given in Exercise 7.8.

Example 7.11. In Chapter 3, the *contours* of the **Ising model** on $\mathbb{Z}^2$ were defined as connected components of line segments (actually, edges of the dual lattice) separating + and − spins. Using the notations for contours adopted in Chapter 3, the relative Hamiltonian (7.10) can be expressed as

$$\mathscr{H}_{\Phi^0}(\omega\,|\,\eta^\pm)=2\Big|\big\{\{i,j\}\in\mathscr{E}_{\mathbb{Z}^2}\,:\,\omega_i\ne\omega_j\big\}\Big|=2\sum_{i=1}^n|\gamma_i|\,. \tag{7.15}$$

Since the ground states $\eta^\#$ are constant, we have $r_*=1$. The difference between the corresponding set $\Gamma(\omega)$ and the contours γ_i is that $\Gamma(\omega)$ is a *thick* object, made of vertices of $\mathbb{Z}^2$ rather than edges of the dual lattice; see Figure 7.4.

Note that, by construction, $\Gamma(\omega)$ is the union of translates of $\mathsf{B}(1)$ centered at vertices $i\in\mathbb{Z}^2$ located at Euclidean distance at most $\sqrt{2}/2$ from a contour. Since

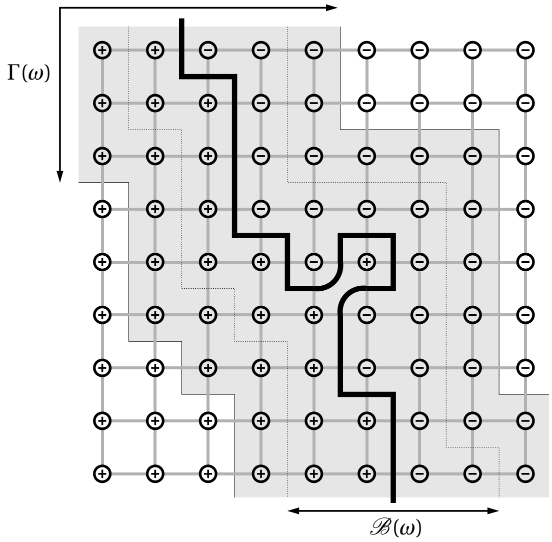


Figure 7.4: A portion of a configuration of the Ising model on $\mathbb{Z}^2$. The thick black line on the dual lattice, that separates + and − spins, is what was called a contour in Chapter 3. The set $\mathscr{B}(\omega)$ of vertices which are neither +- nor −-correct is delimited by the dotted line, and finally the shaded region represents the thickened boundary $\Gamma(\omega) \supset \mathscr{B}(\omega)$.

the total number of such vertices is at most twice the total length of the contours, we have

$$|\Gamma(\omega)| \le 2|\mathsf{B}(1)| \sum_{i=1}^{n} |\gamma_i|.$$

We therefore see that Peierls' condition is satisfied, $\mathscr{H}_{\Phi^0}(\omega\,|\,\eta^\pm) \ge \rho|\Gamma(\omega)|$, with a Peierls constant given by $\rho = |\mathsf{B}(1)|^{-1} = 1/9$. ⋄

Example 7.12. For the **Blume–Capel model** on $\mathbb{Z}^d$, we also have $r_* = 1$. First, since $(\omega_i - \omega_j)^2 \ge 1$ when $\omega_i \ne \omega_j$, we get that

$$\mathscr{H}_{\Phi^0}(\omega\,|\,\eta^\#) \ge \Big|\big\{\{i,j\} \in \mathscr{E}_{\mathbb{Z}^d} : \omega_i \ne \omega_j\big\}\Big|, \tag{7.16}$$

for each ground state $\eta^\# \in \{\eta^+, \eta^0, \eta^-\}$. Then, since we clearly have

$$\Gamma(\omega) \subset \bigcup_{\substack{\{i,j\}\in\mathscr{E}_{\mathbb{Z}^d}\\ \omega_i \ne \omega_j}} (i + \mathsf{B}(1)) \cup (j + \mathsf{B}(1)) \subset \bigcup_{\substack{\{i,j\}\in\mathscr{E}_{\mathbb{Z}^d}\\ \omega_i \ne \omega_j}} (i + \mathsf{B}(2))$$

and thus

$$|\Gamma(\omega)| \le \Big|\big\{\{i,j\} \in \mathscr{E}_{\mathbb{Z}^d} : \omega_i \ne \omega_j\big\}\Big|\,|\mathsf{B}(2)|,$$

it follows that Peierls' condition also holds in this case, with $\rho = |\mathsf{B}(2)|^{-1} = 5^{-d}$. ⋄

Exercise 7.4. *1. Show that*

$$0 \le \psi(\Phi) - (-\underline{e}_\Phi) \le \beta^{-1} \log|\Omega_0|.$$

In particular, $\lim_{\beta\to\infty}\psi(\Phi) = -\underline{e}_\Phi$. Hint: *Choose* $\eta \in g^{\rm per}(\Phi)$*, and start by writing* $\mathscr{H}_{\Lambda;\Phi}(\omega) = \mathscr{H}_{\Lambda;\Phi}(\eta) + \{\mathscr{H}_{\Lambda;\Phi}(\omega) - \mathscr{H}_{\Lambda;\Phi}(\eta)\}$.

2. Assuming now that Φ *satisfies Peierls' condition (with constant* ρ*), show that*

$$0 \le \psi(\Phi) - (-\underline{e}_\Phi) \le |\Omega_0|\beta^{-1} e^{-\beta\rho}.$$

7.2.2 m-potentials

Determining the set of ground states associated to a general potential Φ, as well as checking the validity of Peierls' condition, can be very difficult. [3] Ideally, one would like to do that by checking a finite set of *local* conditions.

Let us define, for each $B \Subset \mathbb{Z}^d$,

$$\phi_B \stackrel{\rm def}{=} \min_\omega \Phi_B(\omega)$$

and set

$$g_m(\Phi) \stackrel{\rm def}{=} \{\omega \in \Omega : \Phi_B(\omega) = \phi_B, \forall B \Subset \mathbb{Z}^d\}.$$

If $g_m(\Phi) \neq \varnothing$, that is, if there exists at least one configuration which minimizes locally each Φ_B, then Φ is called an **m-potential**. We also let $g_m^{\rm per}(\Phi) \stackrel{\rm def}{=} g_m(\Phi) \cap \Omega^{\rm per}$.

Lemma 7.13. *1.* $g_m(\Phi) \subset g(\Phi)$.

2. If $g_m^{\rm per}(\Phi) \neq \varnothing$*, then* $g_m^{\rm per}(\Phi) = g^{\rm per}(\Phi)$.

3. If $0 < |g_m(\Phi)| < \infty$*, then* $g_m(\Phi) = g_m^{\rm per}(\Phi) = g^{\rm per}(\Phi)$*, and* Φ *satisfies Peierls' condition.*

Proof. The first claim is immediate and the second one follows from Lemma 7.4.

For the third claim, observe that $g_m(\Phi)$ is left invariant by any translation of the lattice, in the sense that $\omega \in g_m(\Phi)$ implies $\theta_i\omega \in g_m(\Phi)$ for all $i \in \mathbb{Z}^d$. Therefore, if $g_m(\Phi)$ is finite, all its elements must be periodic. Using the second claim yields $g_m(\Phi) = g_m^{\rm per}(\Phi) = g^{\rm per}(\Phi)$.

Let us now verify that Peierls' condition is satisfied when $0 < |g_m(\Phi)| < \infty$. We first claim that there exists $r \in (0,\infty)$ such that, for any configuration ω for which the vertex $i \in \mathbb{Z}^d$ is not correct, there exists $B \subset i + \mathsf{B}(r)$ such that $\Phi_B(\omega) \neq \phi_B$. Accepting this claim for the moment, the conclusion immediately follows: indeed, one can then set $\epsilon \stackrel{\rm def}{=} \min\{\Phi_B(\omega) - \phi_B : \Phi_B(\omega) > \phi_B, B \subset i + \mathsf{B}(r), \omega \in \Omega \text{ incorrect at } i\} > 0$. Observe that, by translation invariance of Φ, ϵ does not depend on i. We can then write, for any $\eta \in g^{\rm per}(\Phi)$ and any $\omega \stackrel{\infty}{=} \eta$,

$$\mathscr{H}_\Phi(\omega\,|\,\eta) = \sum_{B\Subset\mathbb{Z}^d} \{\Phi_B(\omega) - \phi_B\} \ge \epsilon(2r+1)^{-d}|\mathscr{B}(\omega)|,$$

which shows that Peierls' condition is indeed satisfied.

We thus only need to establish the claim above. Let ω be some configuration such that i is incorrect. We claim that there exists $r' \in (0,\infty)$ such that, for any configuration ω' coinciding with ω on $i+\mathsf{B}(r_*)$, there exists $B\subset i+\mathsf{B}(r')$ such that $\Phi_B(\omega)\neq\phi_B$. (Note that this immediately implies the desired claim, since there are only finitely many possible configurations on $i+\mathsf{B}(r_*)$.) Let us assume the contrary: there exists a sequence of configurations $\omega^{(n)}$, all coinciding with ω on $i+\mathsf{B}(r_*)$, such that $\Phi_B(\omega^{(n)})=\phi_B$ for all $B\subset i+\mathsf{B}(n)$. By sequential compactness of the set Ω (see Proposition 6.20), we can extract a subsequence converging to some configuration ω_* still coinciding with ω on $i+\mathsf{B}(r_*)$ and such that $\Phi_B(\omega_*)=\phi_B$ for all $B\Subset\mathbb{Z}^d$. But this would mean that $\omega_*\in g_m(\Phi)=g^{\mathrm{per}}(\Phi)$, which would contradict the fact that i is incorrect. □

Example 7.14. For the Ising model, Φ^0 (defined in (7.9)) is an m-potential, since the only sets involved are the pairs of nearest-neighbors, $B=\{i,j\}\in\mathscr{E}_{\mathbb{Z}^d}$, and the associated ${\Phi^0}_B(\omega)=-\omega_i\omega_j$ is minimized by taking either $\omega_i=\omega_j=+1$, or $\omega_i=\omega_j=-1$. Lemma 7.13 thus guarantees, as we already knew, that $g^{\mathrm{per}}(\Phi)=g_m(\Phi)=\{\eta^+,\eta^-\}$. ◇

Exercise 7.5. *Study the periodic ground states of the* ***nearest-neighbor Ising antiferromagnet****, in which* $\Omega_0=\{\pm1\}$ *and, for* $h\in\mathbb{R}$,

$$
{\Phi^0}_B(\omega)\stackrel{\text{def}}{=}\begin{cases}-h\omega_i & \text{if } B=\{i\},\\ \omega_i\omega_j & \text{if } B=\{i,j\},\ i\sim j,\\ 0 & \text{otherwise.}\end{cases}
$$

Exercise 7.6. *Consider the modification of the Ising model in* (7.1)*, with* ϵ *sufficiently small, fixed. Study the ground states of that model, as a function of* h*. In particular: for which values of* h *are there two ground states?*

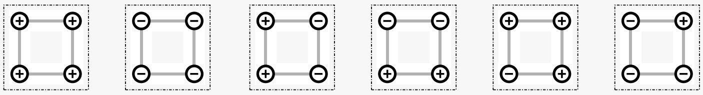
The following exercise shows that it is sometimes possible to find an equivalent potential which is an m-potential, when the original one is not. (However, this is *not* always possible.)

Exercise 7.7. *Let* $\mathscr{T}$ *denote the set of all nearest-neighbor edges of* $\mathbb{Z}^2$*, to which are added all translates of the edge* $\{0,i_c\}$*, where* $i_c=(1,1)$*. Let* $\Omega_0\stackrel{\text{def}}{=}\{\pm1\}$ *and consider the potential* $\Phi=\{\Phi_B\}$ *of the* ***Ising antiferromagnet on the triangular lattice****, defined by*

$$
\Phi_B(\omega)\stackrel{\text{def}}{=}\begin{cases}\omega_i\omega_j & \text{if } B=\{i,j\}\in\mathscr{T},\\ 0 & \text{otherwise.}\end{cases}
$$

1. *Check that* Φ *is not an* m*-potential.*
2. *Construct an* m*-potential* $\widetilde{\Phi}$*, physically equivalent to* Φ*.* Hint: *you can choose it such that* $\widetilde{\Phi}_T>0$ *if and only if* T *is a triangle,* $T=\{i,j,k\}$*, with* $\{i,j\},\{j,k\},\{k,i\}\in\mathscr{T}$*.*
3. *Deduce that* $\widetilde{\Phi}$ *(and thus* Φ*) has an infinite number of periodic ground states.*

Exercise 7.8. *Consider the model on $\mathbb{Z}^2$ with $\Omega_0 \stackrel{\text{def}}{=} \{\pm 1\}$ and in which ${\Phi^0}_B \not\equiv 0$ only if $B = \{i, j, k, l\}$ is a square plaquette (see figure below). If ω coincides, on the plaquette B, with one of the following configurations,*



then ${\Phi^0}_B(\omega) = \alpha$. Otherwise, ${\Phi^0}_B(\omega) = \delta$, with $\delta > \alpha$. Find the periodic ground states of Φ^0, and give some examples of non-periodic ground states. Then, show that Peierls' condition is not satisfied.

7.2.3 Lifting the degeneracy

Consider a system with interactions Φ^0 and a finite set of periodic ground states $g^{\text{per}}(\Phi^0)$, at very low temperature. Using one of the ground states η as a boundary condition, one might wonder whether η is *stable* in the thermodynamic limit, in the sense that typical configurations under the corresponding infinite volume Gibbs measure coincide with η with only sparse, local deviations.

In the Ising model on $\mathbb{Z}^d$, $d \geq 2$, this was the case for both η^+ and η^-. In more general situations, in particular in the absence of a symmetry relating all the elements of $g^{\text{per}}(\Phi^0)$, this issue is much more subtle.

To analyze this problem, we will first introduce a family of *external fields* which will be used to *lift* the degeneracy of the ground states, in the sense that, given any subset $g \subset g(\Phi^0)$, we can tune these external fields to obtain a potential whose set of periodic ground states is given by g. Eventually, these external fields will allow us to prepare the system in the desired Gibbs state and to drive the system from one phase to the other.

To lift the degeneracy, we *perturb* Φ^0 by considering a new potential Φ of the form

$$\Phi = \Phi^0 + W\,,$$

where $W = \{W_B\}_{B \Subset \mathbb{Z}^d}$ is the *perturbation potential.* We first verify that the perturbation, when small enough, does not lead to the appearance of new ground states.

Lemma 7.15. *If Φ^0 satisfies Peierls' condition with Peierls' constant $\rho > 0$ and if $\|W\| \leq \rho/4$, then $g^{\text{per}}(\Phi^0 + W) \subset g^{\text{per}}(\Phi^0)$.*

Proof. Assume that $g^{\text{per}}(\Phi^0) = \{\eta^1, \ldots, \eta^m\}$. Let r_* and $\mathscr{P}$ be as before. Fix some $\omega \in \Omega^{\text{per}}$, and let $[\Gamma](\omega)$ be the set of boxes $b \in \mathscr{P}$ whose intersection with $\Gamma(\omega)$ is non-empty. Again, by Lemma 7.8, boxes b not contained in $[\Gamma](\omega)$ contain only correct vertices, all of the same type. Let therefore $[\Pi_\#](\omega)$, $\# \in \{1, 2, \ldots, m\}$, be the union of those boxes containing only #-correct vertices. Then, let

$$\pi_\#(\omega) \stackrel{\text{def}}{=} \lim_{n\to\infty} \frac{|[\Pi_\#](\omega) \cap \mathrm{B}(n)|}{|\mathrm{B}(n)|}\,, \quad \gamma(\omega) \stackrel{\text{def}}{=} \lim_{n\to\infty} \frac{|[\Gamma](\omega) \cap \mathrm{B}(n)|}{|\mathrm{B}(n)|}\,.$$

Observe that $\gamma(\omega) + \sum_{\#=1}^m \pi_\#(\omega) = 1$ and that $\omega \in g^{\text{per}}(\Phi^0)$ if and only if $\gamma(\omega) = 0$. We will show below that, when $\|W\|$ is sufficiently small,

$$\mathrm{e}_\Phi(\omega) - \mathrm{e}_\Phi(\eta) \geq \sum_{\#=1}^m \pi_\#(\omega)[\mathrm{e}_W(\eta^\#) - \mathrm{e}_W(\eta)] + \tfrac{\rho}{2}\gamma(\omega)\,, \tag{7.17}$$

for all $\eta \in g^{\mathrm{per}}(\Phi^0)$. Assuming this is true, let us take some $\eta \in g^{\mathrm{per}}(\Phi^0)$ for which $\mathsf{e}_W(\eta) = \min_\# \mathsf{e}_W(\eta^\#)$. If $\omega \in g^{\mathrm{per}}(\Phi)$, then in particular $\mathsf{e}_\Phi(\omega) \le \mathsf{e}_\Phi(\eta)$ by Lemma 7.4. So (7.17) gives $\gamma(\omega) = 0$, that is, $\omega \in g^{\mathrm{per}}(\Phi^0)$.

To show (7.17), we start by writing

$$\mathscr{H}_{\mathsf{B}(n);\Phi}(\omega) - \mathscr{H}_{\mathsf{B}(n);\Phi}(\eta) = \big\{\mathscr{H}_{\mathsf{B}(n);\Phi^0}(\omega) - \mathscr{H}_{\mathsf{B}(n);\Phi^0}(\eta)\big\} + \big\{\mathscr{H}_{\mathsf{B}(n);W}(\omega) - \mathscr{H}_{\mathsf{B}(n);W}(\eta)\big\}. \tag{7.18}$$

On the one hand, proceeding as in (7.8),

$$\begin{aligned}\mathscr{H}_{\mathsf{B}(n);\Phi^0}(\omega) - \mathscr{H}_{\mathsf{B}(n);\Phi^0}(\eta) &= \mathscr{H}_{\Phi^0}(\omega^{(n)}|\eta) + O(|\partial^{\mathrm{ex}}\mathsf{B}(n)|)\\ &\ge \rho|\Gamma(\omega^{(n)})| + O(|\partial^{\mathrm{ex}}\mathsf{B}(n)|)\\ &= \rho|\Gamma(\omega)\cap\mathsf{B}(n)| + O(|\partial^{\mathrm{ex}}\mathsf{B}(n)|).\end{aligned}$$

On the other hand, we can decompose

$$\mathscr{H}_{\mathsf{B}(n);W}(\omega) = \sum_{i\in[\Gamma](\omega)\cap\mathsf{B}(n)} u_{i;W}(\omega) + \sum_{\#=1}^{m}\sum_{i\in[\Pi_\#](\omega)\cap\mathsf{B}(n)} u_{i;W}(\omega) + O(|\partial^{\mathrm{ex}}\mathsf{B}(n)|).$$

The first sum can be bounded by

$$\Big|\sum_{i\in[\Gamma](\omega)\cap\mathsf{B}(n)} u_{i;W}(\omega)\Big| \le \big|[\Gamma](\omega)\cap\mathsf{B}(n)\big|\,\|W\|.$$

For the second one, using (7.13),

$$\begin{aligned}\sum_{i\in[\Pi_\#](\omega)\cap\mathsf{B}(n)} u_{i;W}(\omega) &= \sum_{i\in[\Pi_\#](\omega)\cap\mathsf{B}(n)} u_{i;W}(\eta^\#)\\ &= \big|[\Pi_\#](\omega)\cap\mathsf{B}(n)\big|\mathsf{e}_W(\eta^\#) + O(|\partial^{\mathrm{ex}}\mathsf{B}(n)|).\end{aligned}$$

The other Hamiltonian is decomposed as follows:

$$\begin{aligned}\mathscr{H}_{\mathsf{B}(n);W}(\eta) &= |\mathsf{B}(n)|\mathsf{e}_W(\eta) + O(|\partial^{\mathrm{ex}}\mathsf{B}(n)|)\\ &\le \|W\|\big|[\Gamma](\omega)\cap\mathsf{B}(n)\big| + \sum_{\#=1}^{m}\big|[\Pi_\#](\omega)\cap\mathsf{B}(n)\big|\mathsf{e}_W(\eta) + O(|\partial^{\mathrm{ex}}\mathsf{B}(n)|).\end{aligned}$$

Inserting these estimates in (7.18), dividing by $|\mathsf{B}(n)|$, bounding $\|W\| \le \rho/4$ and taking the limit $n \to \infty$ yields (7.17). □

The perturbation of Φ^0 will contain a certain number of parameters (which will play a role analogous to that of the magnetic field in the Ising model), which will allow us to *lift the degeneracy of the ground states of* Φ^0. This means that, if $|g^{\mathrm{per}}(\Phi^0)| = m$, we will need the perturbation W to contain $m-1$ parameters and it should be possible to tune the latter in order for $g^{\mathrm{per}}(\Phi^0 + W)$ to be an arbitrary subset of $g^{\mathrm{per}}(\Phi^0)$. This will be best understood with some examples.

Example 7.16. The degeneracy of the potential Φ^0 of the **Ising model** can be lifted by introducing a magnetic field h and by considering the perturbation $W = \{W_B\}$ defined by

$$W_B(\omega) = \begin{cases} -h\omega_i & \text{if } B = \{i\},\\ 0 & \text{otherwise.}\end{cases}$$

Lemma 7.15 guarantees that $g^{\mathrm{per}}(\Phi) = g^{\mathrm{per}}(\Phi^0 + W) \subset \{\eta^+, \eta^-\}$ when $\|W\| = |h|$ is sufficiently small. But, since the energy densities are given, for all h, by

$$\mathsf{e}_\Phi(\eta^\pm) = -d \mp h\,,$$

we get that $\mathsf{e}_\Phi(\eta^+) < \mathsf{e}_\Phi(\eta^-)$ when $h > 0$, $\mathsf{e}_\Phi(\eta^+) > \mathsf{e}_\Phi(\eta^-)$ when $h < 0$. Therefore, we can describe $g^{\mathrm{per}}(\Phi^0 + W)$ for all h (not only when $|h|$ is small):

$$g^{\mathrm{per}}(\Phi^0 + W) = \begin{cases} \{\eta^+\} & \text{if } h > 0\,, \\ \{\eta^+, \eta^-\} & \text{if } h = 0\,, \\ \{\eta^-\} & \text{if } h < 0\,. \end{cases} \qquad \diamond$$

Example 7.17. In the case of the **Blume–Capel model**, two parameters are necessary to lift the degeneracy. We denote the latter by h and λ, and consider the perturbation $W = \{W_B\}_{B \Subset \mathbb{Z}^d}$ defined by

$$W_B(\omega) \stackrel{\text{def}}{=} \begin{cases} -h\omega_i - \lambda\omega_i^2 & \text{if } B = \{i\}\,, \\ 0 & \text{otherwise.} \end{cases} \tag{7.19}$$

By Lemma (7.15), we know that $g^{\mathrm{per}}(\Phi) \subset \{\eta^+, \eta^0, \eta^-\}$ when $\|W\| = |h| + |\lambda|$ is sufficiently small. The energy densities are given by

$$\mathsf{e}_\Phi(\eta^\pm) = \mp h - \lambda\,, \quad \mathsf{e}_\Phi(\eta^0) = 0\,, \tag{7.20}$$

and the periodic ground states are obtained by studying $\min_\# \mathsf{e}_\Phi(\eta^\#)$ as a function of (λ, h). Let us thus define the regions $\mathscr{U}^+, \mathscr{U}^0, \mathscr{U}^-$, by

$$\mathscr{U}^\# \stackrel{\text{def}}{=} \big\{(\lambda, h) : \mathsf{e}_\Phi(\eta^\#) = \min_{\#'} \mathsf{e}_\Phi(\eta^{\#'})\big\}\,. \tag{7.21}$$

The interior of these regions determines the values of (λ, h) for which there is a unique ground state. Except at $(0,0)$, at which the three ground states coexist, two periodic ground states coexist on the boundaries of these regions, which are unions of lines,

$$\mathscr{L}^{\#\#'} \stackrel{\text{def}}{=} \mathscr{U}^\# \cap \mathscr{U}^{\#'}\,.$$

These are given explicitly by

$$\mathscr{L}^{+-} \stackrel{\text{def}}{=} \{(\lambda, h) : h = 0, \lambda \geq 0\}\,,$$
$$\mathscr{L}^{-0} \stackrel{\text{def}}{=} \{(\lambda, h) : h = \lambda, \lambda \leq 0\}\,,$$
$$\mathscr{L}^{+0} \stackrel{\text{def}}{=} \{(\lambda, h) : h = -\lambda, \lambda \leq 0\}\,.$$

Altogether, we recover the **zero-temperature** phase diagram already depicted on the left of Figure 7.2. ◇

In the following exercise, we see that it is always possible to lift the degeneracy.

Exercise 7.9. *Suppose that $g^{\mathrm{per}}(\Phi^0) = \{\eta^1, \ldots, \eta^m\}$. Provide a collection of potentials $W^1, \ldots, W^{m-1}$ such that, for all $I \subset \{1, \ldots, m\}$, there exist $\lambda^1, \ldots, \lambda^{m-1}$ such that $g^{\mathrm{per}}(\Phi^0 + \sum_{i=1}^{m-1} \lambda_i W^i) = \{\eta^i, i \in I\}$.*

7.2.4 A glimpse of the rest of this chapter

Let us consider a model with potential $\Phi^{\underline{\lambda}} \stackrel{\text{def}}{=} \Phi^0 + \sum_{i=1}^{m-1} \lambda_i W^i$, where the W^i are potentials lifting the degeneracy of the periodic ground states $\eta^1,\dots,\eta^m$ of Φ^0, as explained in the previous section, and $\underline{\lambda} \stackrel{\text{def}}{=} (\lambda_i)_{1\le i\le m-1} \in \mathbb{R}^{m-1}$. We can then construct the zero-temperature phase diagram, by specifying $g^{\text{per}}(\Phi^{\underline{\lambda}})$ for each values of the parameters $\underline{\lambda}$. This phase diagram thus consists of $(m-1)$-dimensional regions with a single periodic ground state, $(m-2)$-dimensional regions in which there are exactly two periodic ground states, etc.

Alternatively, notice that the energy density $\underline{\lambda} \mapsto \underline{\mathrm{e}}_{\Phi^{\underline{\lambda}}}$ of the ground state is a piecewise linear function of $\underline{\lambda}$. The zero-temperature phase diagram characterizes the points $\underline{\lambda}$ at which $\underline{\mathrm{e}}_{\Phi^{\underline{\lambda}}}$ fails to be differentiable.

Our goal in the rest of this chapter is to extend this construction to small positive temperatures. More precisely, we will prove that, in the limit $\beta \to \infty$, the set of values at which $\underline{\lambda} \mapsto \psi_\beta(\underline{\lambda})$ is non-differentiable converges to the corresponding set at which $\underline{\mathrm{e}}_{\Phi^{\underline{\lambda}}}$ fails to be differentiable.

This will be achieved by constructing C^1 functions, $\widehat{\psi}^1_\beta(\underline{\lambda}),\dots,\widehat{\psi}^m_\beta(\underline{\lambda})$, such that the following holds:

1. $\psi_\beta(\underline{\lambda}) = \max_i \widehat{\psi}^i_\beta(\underline{\lambda})$;
2. $\lim_{\beta\to\infty} \widehat{\psi}^i_\beta(\underline{\lambda}) = -\mathrm{e}_{\Phi^{\underline{\lambda}}}(\eta^i)$ for all $i \in \{1,\dots,m\}$;
3. $\lim_{\beta\to\infty} \dfrac{\partial \widehat{\psi}^i_\beta(\underline{\lambda})}{\partial \lambda_j} = -\dfrac{\partial \mathrm{e}_{\Phi^{\underline{\lambda}}}(\eta^i)}{\partial \lambda_j}$, for all $i \in \{1,\dots,m\}$, $j \in \{1,\dots,m-1\}$.

In addition, we will see that the only periodic extremal Gibbs measures at $\underline{\lambda}$ are precisely those obtained by taking the thermodynamic limit with boundary condition η^i for values of i such that $\psi_\beta(\underline{\lambda}) = \widehat{\psi}^i_\beta(\underline{\lambda})$.

Each $\widehat{\psi}^i_\beta$ is called a *truncated pressure*. It is obtained from the partition function with boundary condition η^i by adding the constraint that only "small" (in a sense to be made precise below) excitations are allowed. For certain values of the parameters $\underline{\lambda}$, the excitations turn out to be always small and the truncated pressure coincides with the usual pressure; for others, however, the constraint artificially stabilizes the boundary condition and yields a different, strictly smaller, truncated pressure.

7.2.5 From finite-range interactions to interactions of range one

In Section 7.4, we will initiate the low-temperature analysis of systems with a finite number of periodic ground states, which satisfy Peierls' condition. This analysis will rely on the contour description of these systems, which we expose in detail in the next section.

It turns out that the contour description is considerably simplified if one assumes that the potential Φ under consideration has range 1. Fortunately, any model with a single-spin space Ω_0 and a potential Φ of range $r(\Phi) > 1$ can be mapped onto another model with a potential $\hat{\Phi}$ of range 1, at the cost of introducing a larger single-spin space $\hat{\Omega}_0$, such that the two models have the same pressure.

Earlier, the nuisance of having ground states with different periods was mitigated by considering the boxes $b_k \in \mathscr{P}$ with sidelength r_*, and this can be used further as follows. Assume that $\Phi = \Phi^0 + W$ has range $r(\Phi)$ and that Φ^0 has a finite set of periodic ground states. There are $N \stackrel{\text{def}}{=} |\Omega_0|^{|\mathrm{B}(r_*)|}$ possible configurations inside each of the boxes b_k, and those configurations can be encoded into a new spin variable $\hat{\omega}_k$ taking values in $\hat{\Omega}_0 \stackrel{\text{def}}{=} \{1, 2, \dots, N\}$. Clearly, the set of configurations $\omega = (\omega_i)_{i \in \mathbb{Z}^d} \in \Omega$ is in one-to-one correspondence with the set of configurations $\hat{\omega} = (\hat{\omega}_k)_{k \in \mathbb{Z}^d} \in \hat{\Omega} \stackrel{\text{def}}{=} \hat{\Omega}_0^{\mathbb{Z}^d}$.

By the choice of r_*, it is clear that a spin $\hat{\omega}_k$ only interacts with spins $\hat{\omega}_{k'}$ at distance $d_\infty(k, k') \le 1$. Let us determine the corresponding potential. Denote by $\hat{B}$ a generic union of boxes b_k of diameter at most $2r_*$. For each set $B \Subset \mathbb{Z}^d$ contributing to the original Hamiltonian, let

$$N_B \stackrel{\text{def}}{=} \left|\{\hat{B} : \hat{B} \supset B\}\right|.$$

The terms of the formal Hamiltonian can be rearranged as follows:

$$\sum_B \Phi_B(\omega) = \sum_{\hat{B}} \Big\{ \sum_{B \subset \hat{B}} \frac{1}{N_B} \Phi_B(\omega) \Big\}.$$

We are led to defining the **rescaled potential** as

$$\hat{\Phi}_{\hat{B}}(\hat{\omega}) \stackrel{\text{def}}{=} \sum_{B \subset \hat{B}} \frac{1}{N_B} \Phi_B(\omega).$$

Clearly, all the information about the original model can be recovered from the rescaled model (with $\hat{\Omega}$ and $\hat{\Phi}$); in particular, they have the same pressure (up to a multiplicative constant).

By construction, the rescaled measure $\hat{\Phi}$ has range $r(\hat{\Phi}) = 1$ (as measured on the rescaled lattice $r_* \mathbb{Z}^d$). Of course, analyzing the set of ground states and the validity of Peierls' condition for Φ is equivalent to accomplishing these tasks for the rescaled model. Besides having interactions of range 1, this reformulation of the model presents the advantage that, now, the ground states correspond to *constant* configurations on $\hat{\Omega}$.

Exercise 7.10. *Assume that the original potential Φ^0 satisfies Peierls' condition with constant $\rho > 0$. Show that Peierls' condition still holds for the rescaled model ($\omega \mapsto \hat{\omega}$, $\Phi \mapsto \hat{\Phi}$) and estimate the corresponding constant.*

Since the above construction can always be implemented, we assume, from now on, that the model has been suitably formulated so as to have range 1 and a finite set of constant ground states. In this way, the analysis will become substantially simpler, without incurring any loss of generality.

7.2.6 Contours and their labels

Let us therefore consider a potential $\Phi = \Phi^0 + W$ with $r(\Phi^0) = 1$ and $r(W) \le 1$ and such that the ground states of Φ^0,

$$g^{\text{per}}(\Phi^0) = \{\eta^1, \dots, \eta^m\},$$

are all constant. We also assume that the parameters contained in W completely lift the degeneracy of the ground state. Since $r(\Phi) = 1$, the set $\Gamma(\omega)$ in (7.14) is defined using $r_* = 1$.

Since W does not introduce any new ground states (Lemma 7.15), we may expect, roughly, a typical configuration ω of the model associated to Φ to display, at low temperature, only small local deviations away from one of the ground states $\eta^\#$. We thus start an analysis of the perturbed model in terms of the decomposition of $\Gamma(\omega)$ into *contours*, which separate regions on which the ground states $\{\eta^1,\dots,\eta^m\}$ are seen. This is very similar to what was done when studying the low-temperature Ising model in Chapter 3.

Before pursuing, let us define the notion of connectedness used in the rest of the chapter, based on the use of the distance $d_\infty(\cdot,\cdot)$: $A \subset \mathbb{Z}^d$ is **connected** if for all pair $j, j' \in A$ there exists a sequence $i_1 = j, i_2, \dots, i_{n-1}, i_n = j'$ such that $i_k \in A$ for all $k = 1,\dots,n$, and $d_\infty(i_k, i_{k+1}) = 1$. A connected component $A' \subset A$ is **maximal** if any set $B \neq A'$ such that $A' \subset B \subset A$ is necessarily disconnected.

When $\omega \overset{\infty}{=} \eta$ for some $\eta \in g^{\mathrm{per}}(\Phi)$, the set $\Gamma(\omega)$ is bounded and can be decomposed into maximal connected components:

$$\Gamma(\omega) = \{\overline{\gamma}_1,\dots,\overline{\gamma}_n\}\,.$$

For each component $\overline{\gamma} \in \Gamma(\omega)$, let $\omega_{\overline{\gamma}}$ denote the restriction of ω to $\overline{\gamma}$. The configuration $\omega_{\overline{\gamma}}$ should be considered as being part of the information contained in the component:

Definition 7.18. *Each pair $\gamma \overset{\text{def}}{=} (\overline{\gamma}, \omega_{\overline{\gamma}})$ is called a **contour** of ω; $\overline{\gamma}$ is the **support** of γ.*

The support of a contour γ splits $\mathbb{Z}^d$ into a finite number of maximal connected components (see Figure 7.5):

$$\overline{\gamma}^{\mathrm{c}} = \{A_0, A_1, \dots, A_k\}\,. \tag{7.22}$$

Exactly one of the components of $\overline{\gamma}^{\mathrm{c}}$ is unbounded; with no loss of generality we can assume it to be A_0. We call it the **exterior** of γ and denote it by $\mathrm{ext}\gamma$.

Let us say that a subset $A \subset \mathbb{Z}^d$ is c-**connected** if A^{c} is connected.

Exercise 7.11. *Show that the subsets $A_0,\dots,A_k$ in the decomposition* (7.22) *are* c-*connected.*

Remember the boundaries $\partial^{\mathrm{in}} A$ and $\partial^{\mathrm{ex}} A$ introduced in (7.3). Although the content of the following lemma might seem intuitively obvious to the reader (at least in low dimensions), we provide a proof in Appendix B.15.

Lemma 7.19. *Consider the decomposition* (7.22)*. For each $j = 0,1,\dots,k$, $\partial^{\mathrm{ex}} A_j$ and $\partial^{\mathrm{in}} A_j$ are connected. Moreover, there exists $\# \in \{1,2,\dots,m\}$, depending on j, such that $\omega_i = \eta^\#_i$ for all $i \in \partial^{\mathrm{ex}} A_j$. We call # the **label of A_j**, and denote it* $\mathrm{lab}(A_j)$.

Consider the decomposition (7.22) of some contour $\gamma = (\overline{\gamma}, \omega_{\overline{\gamma}})$. If the exterior has label $\mathrm{lab}(\mathrm{ext}\gamma) = \#$, we say that γ is of **type** #. The remaining components in (7.22), $A_1,\dots,A_k$, are all bounded and separated from $\mathrm{ext}\gamma$ by $\overline{\gamma}$. We group them according to their type. The **interior of type #** of γ is defined by

$$\mathrm{int}_\# \gamma \overset{\text{def}}{=} \bigcup_{\substack{i \in \{1,\dots,k\}:\\ \mathrm{lab}(A_i) = \#}} A_i\,.$$

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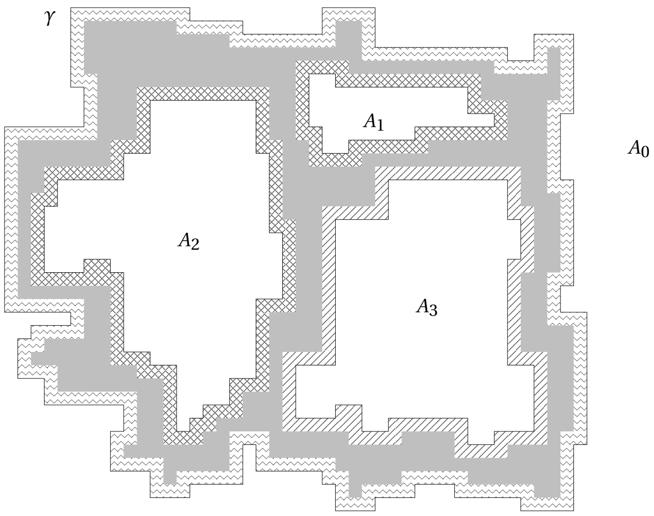

Figure 7.5: A contour γ for which $\overline{\gamma}^{c} = \{A_0, A_1, A_2, A_3\}$. The labels, whose existence is guaranteed by Lemma 7.19, have been pictured using different patterns. The components A_1 and A_2 have the same label. The label of $A_0 = \text{ext}\gamma$ represents the type of γ. This picture shows how the labels induce a corresponding boundary condition on the interior components of γ (which we use later when defining $\mathbf{Z}^{\#}_{\Phi}(\Lambda)$ in (7.24)).

We will also call $\text{int}\gamma \stackrel{\text{def}}{=} \bigcup_{\#=1}^{m} \text{int}_{\#}\gamma$ the **interior** of γ.

The collection of all possible contours of type # is denoted by $\mathscr{C}^{\#}$. Observe that, for each contour $\gamma = (\overline{\gamma}, \omega_{\overline{\gamma}}) \in \mathscr{C}^{\#}$, there exists a configuration that has γ as its unique contour. Namely, extend $\omega_{\overline{\gamma}} = (\omega_i)_{i\in\overline{\gamma}}$ to a configuration on the whole lattice by setting $\omega_i = \eta^{\#}_i$ for $i \in \text{ext}\gamma$ and $\omega_i = \eta^{\#'}_i$ for each $i \in \text{int}_{\#'}\gamma$. For notational convenience, we also denote this new configuration by $\omega_{\overline{\gamma}}$.

The type and labels associated to a contour will play an essential role in next section.

7.3 Boundary conditions and contour models

Notice that if γ, γ' are two contours in a same configuration, and if $\overline{\gamma} \subset \text{int}\gamma'$, then $d_\infty(\overline{\gamma}, (\text{int}\gamma')^{c}) > 1$.

Let $\Lambda \Subset \mathbb{Z}^d$. *From now on, we always assume that* Λ *is* c*-connected.* To define contour models in Λ, it will be convenient to slightly modify the way in which boundary conditions are introduced: this will make it easier to consider the boundary condition induced by a contour on its interior.

Let $\eta^{\#} \in g^{\text{per}}(\Phi^0)$, and $\omega \in \Omega^{\eta^{\#}}_{\Lambda}$. Since it is not guaranteed that $\Gamma(\omega) \subset \Lambda$, we define

$$\Omega^{\#}_{\Lambda} \stackrel{\text{def}}{=} \{\omega \in \Omega^{\eta^{\#}}_{\Lambda} : d_\infty(\Gamma(\omega), \Lambda^{c}) > 1\}.$$

The additional restrictions imposed on the configurations in $\Omega^{\#}_{\Lambda}$ only affect vertices located near the boundary of Λ:

Lemma 7.20. *Let $\omega \in \Omega_\Lambda^{\eta^\#}$. Then, $\omega \in \Omega_\Lambda^\#$ if and only if $\omega_i = \eta_i^\#$ for every vertex $i \in \Lambda$ satisfying $d_\infty(i, \Lambda^c) \le 3$.*

Proof. Let $\omega \in \Omega_\Lambda^{\eta^\#}$ and assume that there exists $i \in \Lambda$, $d_\infty(i,\Lambda^c) \le 3$, such that $\omega_i \neq \eta_i^\#$. This implies that there exists some $j \in i + \mathsf{B}(1)$ with $d_\infty(j, \Lambda^c) \le 2$ which is not #-correct, and so $j \in \mathscr{B}(\omega)$. Since $d_\infty(j + \mathsf{B}(1), \Lambda^c) \le 1$, this implies that $d_\infty(\Gamma(\omega), \Lambda^c) \le 1$ and, thus, $\omega \notin \Omega_\Lambda^\#$.

Conversely, suppose that $\omega_i = \eta_i^\#$ as soon as $d_\infty(i, \Lambda^c) \le 3$. Then, any vertex $i \in \Lambda$ with $d_\infty(i, \Lambda^c) \le 2$ is #-correct. Therefore, vertices which are not correct must satisfy $d_\infty(i, \Lambda^c) \ge 3$ and, thus, $d_\infty(i + \mathsf{B}(1), \Lambda^c) \ge 2$. This implies that $d_\infty(\Gamma(\omega), \Lambda^c) > 1$. □

In order to use contours and their weights for the description of finite systems, it will be convenient to introduce boundary conditions as above, using $\Omega_\Lambda^\#$ instead of $\Omega_\Lambda^{\eta^\#}$. We therefore consider the following Gibbs distributions: for all $\omega \in \Omega_\Lambda^\#$,

$$\mu_{\Lambda;\Phi}^\#(\omega) \stackrel{\text{def}}{=} \frac{e^{-\beta \mathscr{H}_{\Lambda;\Phi}(\omega)}}{\mathbf{Z}_\Phi^\#(\Lambda)}, \tag{7.23}$$

where

$$\mathbf{Z}_\Phi^\#(\Lambda) \stackrel{\text{def}}{=} \sum_{\omega \in \Omega_\Lambda^\#} e^{-\beta \mathscr{H}_{\Lambda;\Phi}(\omega)}. \tag{7.24}$$

It follows from Lemma 7.20 that (remember Exercise 7.1)

$$\lim_{\Lambda \Uparrow \mathbb{Z}^d} \frac{1}{\beta|\Lambda|} \log \mathbf{Z}_\Phi^\#(\Lambda) = \lim_{\Lambda \Uparrow \mathbb{Z}^d} \frac{1}{\beta|\Lambda|} \log \mathbf{Z}_\Phi^{\eta^\#}(\Lambda) = \psi(\Phi).$$

Let us say that Λ is **thin** if $d_\infty(i, \Lambda^c) \le 3$ for all $i \in \Lambda$. It follows from Lemma 7.20 that, whenever Λ is thin, $\Omega_\Lambda^\# = \{\eta^\#\}$ and, thus,

$$\mathbf{Z}_\Phi^\#(\Lambda) = e^{-\beta \mathscr{H}_{\Lambda;\Phi}(\eta^\#)}. \tag{7.25}$$

7.3.1 Extracting the contribution from the ground state

Let us fix a boundary condition $\# \in \{1, 2, \dots, m\}$ and relate the energy of each configuration $\omega \in \Omega_\Lambda^\#$ to the energy of $\eta^\#$:

$$\begin{aligned} \mathscr{H}_{\Lambda;\Phi}(\omega) &= \mathscr{H}_{\Lambda;\Phi}(\eta^\#) + \left\{\mathscr{H}_{\Lambda;\Phi}(\omega) - \mathscr{H}_{\Lambda;\Phi}(\eta^\#)\right\} \\ &= \mathscr{H}_{\Lambda;\Phi}(\eta^\#) + \mathscr{H}_\Phi(\omega \,|\, \eta^\#). \end{aligned} \tag{7.26}$$

One can thus write

$$\mathbf{Z}_\Phi^\#(\Lambda) = e^{-\beta \mathscr{H}_{\Lambda;\Phi}(\eta^\#)} \sum_{\omega \in \Omega_\Lambda^\#} e^{-\beta \mathscr{H}_\Phi(\omega \,|\, \eta^\#)} \stackrel{\text{def}}{=} e^{-\beta \mathscr{H}_{\Lambda;\Phi}(\eta^\#)} \Xi_\Phi^\#(\Lambda). \tag{7.27}$$

Notice that, since each ground state $\eta^\#$ is constant and Φ has range 1,

$$\mathscr{H}_{\Lambda;\Phi}(\eta^\#) = \mathsf{e}_\Phi(\eta^\#)|\Lambda|.$$

Our next goal is to express $\Xi_\Phi^\#(\Lambda)$ as the partition function of a polymer model having the same abstract structure as those of Section 5.2. To this end the contours introduced above will play the role of polymers. Remember, however, that the compatibility condition used in Section 5.2 was *pairwise*. Unfortunately, our contours have labels, and this yields a more complex compatibility condition.

To determine whether a given family of contours is compatible, that is, whether there exists a configuration yielding precisely this family of contours, we need to verify two conditions. The first one is that their supports are disjoint and sufficiently far apart, in a suitable sense; this can of course be expressed as a pairwise condition. However, we must also check that their labels match*, and this condition cannot be verified by only looking at pairs of contours. We illustrate this on Figure 7.6.* ⋄

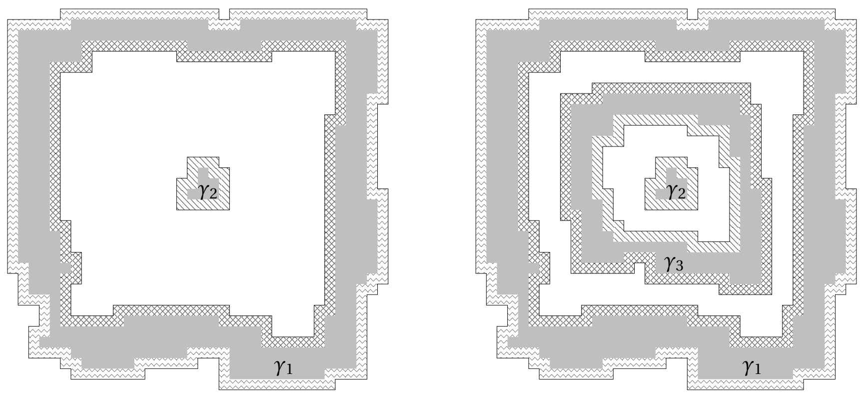

Figure 7.6: Two contours γ_1 and γ_2 (left). γ_2 is of type 1 and satisfies $\overline{\gamma}_2 \subset \text{int}_2\gamma_1$. These two contours can only be part of a configuration if there are other contours correcting the mismatch between the type of γ_2 and the label of the component of γ_1 it is located in. For example (right), there might be a third contour γ_3 of type 2 such that $\overline{\gamma}_3 \subset \text{int}_2\gamma_1$ and $\overline{\gamma}_2 \subset \text{int}_1\gamma_3$. This shows that the compatibility of a family of contours is a global property, which cannot be expressed pairwise.

To deal with this problem, we need to proceed with more care than in (7.27) and express $\Xi^{\#}_{\Phi}(\Lambda)$ as a polymer model in which the polymers are contours *all of the same type* #, for which the compatibility condition becomes purely geometrical, namely having supports which are far apart, in the following sense.

Definition 7.21. *Two contours of the same type, γ_1 and γ_2, are said to be* ***compatible*** *if $d_\infty(\overline{\gamma}_1, \overline{\gamma}_2) > 1$.*

By construction, all contours appearing in a same configuration ω are compatible. The important distinction that must be made among the contours of a configuration is the following:

Definition 7.22. *Let $\omega \in \Omega^{\#}_{\Lambda}$. A contour $\gamma' \in \Gamma(\omega)$ is* ***external*** *if there exist no contour $\gamma \in \Gamma(\omega)$ such that $\overline{\gamma}' \subset \text{int}\gamma$.*

We will group the configurations that contribute to $\mathbf{Z}^{\#}_{\Phi}(\Lambda)$ into families of configurations that have the same set of external contours. If $\Gamma(\omega) \neq \varnothing$, there exists at least one external contour, so let $\Gamma' \subset \Gamma(\omega)$ denote the collection of external contours of ω, which are pairwise compatible by construction. Let then

$$\mathrm{ext} \stackrel{\mathrm{def}}{=} \bigcap_{\gamma'\in\Gamma'} \mathrm{ext}\gamma', \quad \Lambda^{\mathrm{ext}} \stackrel{\mathrm{def}}{=} \Lambda \cap \mathrm{ext}.$$

The important property shared by the external contours of a configuration is that they all have the same type:

Lemma 7.23. *For all* $\omega \in \Omega^{\#}_{\Lambda}$, ext *is connected and* $\omega_i = \eta^{\#}_i$ *for each* $i \in$ ext. *As a consequence, all external contours of* $\Gamma(\omega)$ *are of type* #.

Proof. Let $i', i'' \in$ ext and consider an arbitrary path $i' = i_1, \dots, i_n = i''$, $d_\infty(i_k, i_{k+1}) = 1$. If the path intersects the support of some external contour $\gamma' \in \Gamma(\omega)$, we define $k_- \stackrel{\mathrm{def}}{=} \min\{k : i_k \in \overline{\gamma}'\} - 1$ and $k_+ \stackrel{\mathrm{def}}{=} \max\{k : i_k \in \overline{\gamma}'\} + 1$. Clearly, $\{i_{k_-}, i_{k_+}\} \subset \partial^{\mathrm{in}}\mathrm{ext}\gamma'$. By Lemma 7.19, $\partial^{\mathrm{in}}\mathrm{ext}\gamma'$ is connected. One can therefore modify the path, between i_{k_-} and i_{k_+}, so that it is completely contained in $\partial^{\mathrm{in}}\mathrm{ext}\gamma'$. Since this can be done for each γ', we obtain in the end a path which is contained in each $\partial^{\mathrm{in}}\mathrm{ext}\gamma'$, hence in ext. This shows that ext is connected. The two other claims are immediate consequences. □

Remark 7.24. In this chapter, we always assume that the dimension is at least 2. Nevertheless, we invite the reader to stop and ponder over the peculiarities of the above-defined contours when $d = 1$. ⋄

Now, since Λ is assumed to be c-connected, it can be partitioned into

$$\Lambda = \Lambda^{\mathrm{ext}} \cup \bigcup_{\gamma'\in\Gamma'} \{\overline{\gamma}' \cup \bigcup_{\#'} \mathrm{int}_{\#'}\gamma'\},$$

and we can then rearrange the sum over the sets $B \cap \Lambda \neq \varnothing$, in the Hamiltonian, to obtain:

$$\mathscr{H}_{\Lambda;\Phi}(\omega) = \mathscr{H}_{\Lambda^{\mathrm{ext}};\Phi}(\omega) + \sum_{\gamma'\in\Gamma'} \Big\{ \sum_{B\subset\overline{\gamma}'} \Phi_B(\omega) + \sum_{\#'} \mathscr{H}_{\mathrm{int}_{\#'}\gamma';\Phi}(\omega) \Big\}. \tag{7.28}$$

We have used the fact that the contours are thick, which implies that the components of their complement are at distance larger than the range of Φ (remember Lemma 7.8). Observe that for each $B \subset \overline{\gamma}'$, $\Phi_B(\omega) = \Phi_B(\omega_{\overline{\gamma}'})$.

Let us characterize all configurations $\omega \in \Omega^{\#}_{\Lambda}$ that have the same set of external contours Γ':

1. Since Λ^{ext} does not contain any contours and since $\omega_i = \eta^{\#}_i$ for all $i \in \partial^{\mathrm{ex}}\Lambda^{\mathrm{ext}}$, we must have $\omega_i = \eta^{\#}_i$ for each $i \in \Lambda^{\mathrm{ext}}$. In particular, $\mathscr{H}_{\Lambda^{\mathrm{ext}};\Phi}(\omega) = \mathscr{H}_{\Lambda^{\mathrm{ext}};\Phi}(\eta^{\#})$.
2. Each component of each $\mathrm{int}_{\#'}\gamma'$ has a boundary condition specified by the label of that component, namely $\#'$, and the contours of the configuration on that component must be at distance larger than 1 from γ'. The restrictions for the allowed configurations on $\mathrm{int}_{\#'}\gamma'$ therefore coincide exactly with those of $\Omega^{\#'}_{\mathrm{int}_{\#'}\gamma'}$.

Using (7.28), we thus get, after resumming over the allowed configurations on each component of $\mathrm{int}_{\#'}\overline{\gamma}'$:

$$\mathbf{Z}^{\#}_{\Phi}(\Lambda) = \sum_{\substack{\Gamma' \\ \text{compatible} \\ \text{external}}} e^{-\beta\mathscr{H}_{\Lambda^{\mathrm{ext}};\Phi}(\eta^{\#})} \prod_{\gamma'\in\Gamma'} \Big\{ \exp\big(-\beta \sum_{B\subset\overline{\gamma}'} \Phi_B(\omega_{\overline{\gamma}'})\big) \prod_{\#'} \mathbf{Z}^{\#'}_{\Phi}(\mathrm{int}_{\#'}\gamma') \Big\}. \tag{7.29}$$

Let us define, for each $\gamma \in \mathscr{C}^{\#}$, the **surface energy**

$$\|\gamma\| \stackrel{\text{def}}{=} \sum_{B \subset \overline{\gamma}} \left\{ \Phi_B(\omega_{\overline{\gamma}}) - \Phi_B(\eta^{\#}) \right\}.$$

With this notation, (7.29) can be rewritten as

$$e^{\beta \mathscr{H}_{\Lambda;\Phi}(\eta^{\#})} \mathbf{Z}_{\Phi}^{\#}(\Lambda) = \sum_{\substack{\Gamma' \\ \text{compatible} \\ \text{external}}} \prod_{\gamma' \in \Gamma'} \left\{ e^{-\beta \|\gamma'\|} \prod_{\#'} e^{\beta \mathscr{H}_{\text{int}_{\#'}\gamma';\Phi}(\eta^{\#})} \mathbf{Z}_{\Phi}^{\#'}(\text{int}_{\#'}\gamma') \right\}. \qquad (7.30)$$

Our aim is then to go one step further and consider the external contours contained in each partition function $\mathbf{Z}_{\Phi}^{\#'}(\text{int}_{\#'}\gamma')$ appearing on the right-hand side. Unfortunately, the external contours in $\mathbf{Z}_{\Phi}^{\#'}(\text{int}_{\#'}\gamma')$ are of type $\#'$, and one needs to remove these from the analysis in order to avoid the global compatibility problem mentioned earlier.

In order to only deal with external contours of type #, we will use the following trick [4]: we multiply and divide the product over $\#'$, in (7.30), by the partition functions that involve only the #-boundary condition. That is, we write

$$\prod_{\#'} \mathbf{Z}_{\Phi}^{\#'}(\text{int}_{\#'}\gamma') = \left\{ \prod_{\#'} \frac{\mathbf{Z}_{\Phi}^{\#'}(\text{int}_{\#'}\gamma')}{\mathbf{Z}_{\Phi}^{\#}(\text{int}_{\#'}\gamma')} \right\} \prod_{\#'} \mathbf{Z}_{\Phi}^{\#}(\text{int}_{\#'}\gamma'). \qquad (7.31)$$

This introduces a non-trivial quotient that will be taken care of later, but it has the advantage of making the partition functions $\mathbf{Z}_{\Phi}^{\#}(\text{int}_{\#'}\gamma')$ appear, which all share the same boundary condition #. This means that if one starts again summing over the external contours in $\mathbf{Z}_{\Phi}^{\#}(\text{int}_{\#'}\gamma')$, *these will again be of type #*, as in the first step.

Let us express (7.30) using only the partition functions $\Xi_{\Phi}^{\#}(\cdot)$. Remembering that $e^{\beta \mathscr{H}_{\text{int}_{\#'}\gamma';\Phi}(\eta^{\#})} \mathbf{Z}_{\Phi}^{\#}(\text{int}_{\#'}\gamma') \stackrel{\text{def}}{=} \Xi_{\Phi}^{\#}(\text{int}_{\#'}\gamma')$, (7.30) becomes

$$\Xi_{\Phi}^{\#}(\Lambda) = \sum_{\substack{\Gamma' \\ \text{compatible} \\ \text{external}}} \prod_{\gamma' \in \Gamma'} \left\{ \mathrm{w}^{\#}(\gamma') \prod_{\#'} \Xi_{\Phi}^{\#}(\text{int}_{\#'}\gamma') \right\}, \qquad (7.32)$$

where we introduced, for each $\gamma \in \mathscr{C}^{\#}$, the weight

$$\mathrm{w}^{\#}(\gamma) \stackrel{\text{def}}{=} e^{-\beta\|\gamma\|} \prod_{\#'} \frac{\mathbf{Z}_{\Phi}^{\#'}(\text{int}_{\#'}\gamma)}{\mathbf{Z}_{\Phi}^{\#}(\text{int}_{\#'}\gamma)}. \qquad (7.33)$$

Looking at (7.32), it is clear that we can now repeat the procedure of fixing the external contours for each factor $\Xi_{\Phi}^{\#}(\text{int}_{\#'}\gamma')$, these being all of type #. This process can be iterated and will automatically stop when one reaches contours whose interior is thin (remember the discussion on page 342), since the latter are too small to contain other contours. In this way, we end up with the following contour representation of the partition function:

$$\Xi_{\Phi}^{\#}(\Lambda) = \sum_{\substack{\Gamma \\ \text{compatible}}} \prod_{\gamma \in \Gamma} \mathrm{w}^{\#}(\gamma), \qquad (7.34)$$

where the sum is over collections of contours of type #, in which the compatibility (in the sense of Definition 7.21) is purely geometrical, and can be encoded into

$$\delta(\gamma, \gamma') \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } d_\infty(\overline{\gamma}, \overline{\gamma}') > 1, \\ 0 & \text{otherwise.} \end{cases}$$

This pairwise hard-core interaction is similar to the one encountered in Section 5.7.1 (with a different distance). Notice, however, that the polymers considered here are more complex objects, which contain more information: their support but also the partial configuration $\omega_{\overline{\gamma}}$ (and the labels it induces).

Remark 7.25. It is important to emphasize that a compatible collection of contours contributing to (7.34) is an abstract collection, which *does not correspond*, in general, to the contours of any configuration $\omega \in \Omega^{\#}_{\Lambda}$. ⋄

We have thus managed to express the partition function as a polymer model with a purely geometrical, pairwise compatibility condition. The price we had to pay for that was the introduction of the nontrivial weights $\mathrm{w}^{\#}(\cdot)$. The very nature of the latter suggests an *inductive* analysis. Namely, since $\mathrm{w}^{\#}(\gamma)$ can be written

$$\mathrm{w}^{\#}(\gamma) = e^{-\beta\|\gamma\|} \prod_{\#'} \frac{e^{-\beta \mathscr{H}_{\mathrm{int}_{\#'}\gamma;\Phi}(\eta^{\#'})}}{e^{-\beta \mathscr{H}_{\mathrm{int}_{\#'}\gamma;\Phi}(\eta^{\#})}} \frac{\Xi^{\#'}_{\Phi}(\mathrm{int}_{\#'}\gamma)}{\Xi^{\#}_{\Phi}(\mathrm{int}_{\#'}\gamma)}, \tag{7.35}$$

we see that $\mathrm{w}^{\#}(\gamma)$ depends on the weights of the smaller contours that appear in each $\Xi^{\#'}_{\Phi}(\mathrm{int}_{\#'}\gamma)$ (which are of type $\#' \neq \#$) and in $\Xi^{\#}_{\Phi}(\mathrm{int}_{\#'}\gamma)$.

7.3.2 Representing probabilities involving external contours

Before going further, let us see how the contour models presented above can be used to represent probabilities involving external contours. Remember the definition of $\mu^{\#}_{\Lambda;\Phi}$ in (7.23).

Lemma 7.26. *Let Λ be* c*-connected and let $\{\gamma'_1,\dots,\gamma'_k\}$ be a collection of pairwise compatible contours of type # such that each γ'_i is contained in the exterior of the others, and $d_\infty(\gamma'_i,\Lambda^{\mathrm{c}}) > 1$. Then*

$$\mu^{\#}_{\Lambda;\Phi}\big(\Gamma' \supset \{\gamma'_1,\dots,\gamma'_k\}\big) \leq \prod_{i=1}^{k} \mathrm{w}^{\#}(\gamma'_i). \tag{7.36}$$

Proof. Follows the same steps that started with (7.29). □

Exercise 7.12. *Complete the proof of Lemma 7.26.*

7.4 Phase diagram of the Blume–Capel model

From now on, for the sake of concreteness, we will stick to the Blume–Capel model. As before, the three constant ground states are denoted by $\eta^{\#}$, with $\# \in \{+,0,-\}$. We lift the degeneracy using W defined in (7.19). We continue to omit the dependence on β everywhere. We also drop Φ from the notations and only indicate the dependence on (λ,h) when it is really needed, that is, we write $\mathbf{Z}^{\#}(\Lambda)$ rather than $\mathbf{Z}^{\#}_{\lambda,h}(\Lambda)$, etc. We also write $\mathrm{e}^{\#}$ (or $\mathrm{e}^{\#}(\lambda,h)$ if necessary) instead of $\mathrm{e}_{\Phi}(\eta^{\#})$. With these conventions, (7.27) becomes

$$\mathbf{Z}^{\#}(\Lambda) = e^{-\beta \mathrm{e}^{\#}|\Lambda|}\,\Xi^{\#}(\Lambda).$$

We denote the pressure of the model, defined as in (7.7), by $\psi = \psi(\lambda,h)$.

7.4.1 Heuristics

We will construct the phase diagram by determining the *stable phases* of the system. Loosely speaking, this will consist in the determination, for each choice of boundary condition $\# \in \{+,0,-\}$, of the set of pairs (λ, h) for which a Gibbs measure can be constructed using the thermodynamic limit with boundary condition #, whose typical configurations are small deviations from the ground state $\eta^{\#}$. Eventually, this will be done in Theorem 7.41.

But we will first focus on the pressure, in particular the pressure in a finite volume with the boundary condition #:

$$\frac{1}{\beta|\Lambda|}\log \mathbf{Z}^{\#}(\Lambda) = -\mathrm{e}^{\#} + \frac{1}{\beta|\Lambda|}\log \Xi^{\#}(\Lambda)\,.$$

In a regime where (λ, h) is such that $\mathrm{e}^{\#}$ is minimal among all $\mathrm{e}^{\#'}$, which happens when $(\lambda, h) \in \mathscr{U}^{\#}$, we expect typical configurations to be described by sparse local deviations away from $\eta^{\#}$; these configurations should also be the main contributions to $\mathbf{Z}^{\#}(\Lambda)$, in the sense that $-\mathrm{e}^{\#}$ should be the leading contribution to the pressure, the term $\frac{1}{\beta|\Lambda|}\log \Xi^{\#}(\Lambda)$ representing only *corrections* (for large values of β).

Making this argument rigorous requires having a control over $\log \Xi^{\#}(\Lambda)$; it will involve a detailed analysis of the weights $\mathrm{w}^{\#}(\cdot)$ and will eventually rely on a balance between the fields and an isoperimetric ratio related to the volume and support of the contours. Let us describe how the latter appear.

We know from Theorem 5.4 that $\log \Xi^{\#}(\Lambda)$ admits a convergent cluster expansion, in any finite region Λ, provided that one can find numbers $a(\gamma) \geq 0$ such that (the weights $\mathrm{w}^{\#}(\cdot)$ being real and nonnegative, there is no need for absolute values here and below)

$$\forall \gamma_* \in \mathscr{C}^{\#}, \quad \sum_{\gamma \in \mathscr{C}^{\#}} \mathrm{w}^{\#}(\gamma) e^{a(\gamma)} |\zeta(\gamma, \gamma_*)| \leq a(\gamma_*)\,, \tag{7.37}$$

where $\zeta(\gamma, \gamma_*) \stackrel{\text{def}}{=} \delta(\gamma, \gamma_*) - 1$. As in Section 5.7.1, we observe that $\zeta(\gamma, \gamma_*) \neq 0$ if and only if $\overline{\gamma} \cap [\overline{\gamma}_*] \neq \varnothing$, where $[\overline{\gamma}_*] \stackrel{\text{def}}{=} \{j \in \mathbb{Z}^d : d_\infty(j, \overline{\gamma}_*) \leq 1\}$. This gives

$$\sum_{\gamma \in \mathscr{C}^{\#}} \mathrm{w}^{\#}(\gamma) e^{a(\gamma)} |\zeta(\gamma, \gamma_*)| \leq |[\overline{\gamma}_*]| \sup_{i \in \mathbb{Z}^d} \sum_{\gamma \in \mathscr{C}^{\#} : \overline{\gamma} \ni i} \mathrm{w}^{\#}(\gamma) e^{a(\gamma)}\,.$$

This shows that $a(\gamma) \stackrel{\text{def}}{=} |[\overline{\gamma}]|$ is a natural candidate. Since $|[\overline{\gamma}]| \leq 3^d |\overline{\gamma}|$, (7.37) is satisfied if

$$\sum_{\gamma \in \mathscr{C}^{\#} : \overline{\gamma} \ni 0} \mathrm{w}^{\#}(\gamma) e^{3^d |\overline{\gamma}|} \leq 1\,. \tag{7.38}$$

Clearly, (7.38) can hold only if $\mathrm{w}^{\#}(\gamma)$ decreases exponentially fast with the size of the support of γ. We are thus naturally led to the following notion.

Definition 7.27. *The weight* $\mathrm{w}^{\#}(\gamma)$ *is* τ*-**stable** if*

$$\mathrm{w}^{\#}(\gamma) \leq e^{-\tau|\overline{\gamma}|}\,.$$

Below, in Lemma 7.30, we will show that (7.38) is indeed verified provided that all the weights $\mathrm{w}^{\#}(\gamma)$ are τ-stable (for a sufficiently large value of τ). Of course, this will be true only for certain values of (λ, h). For the moment, let us make a few comments about the difficulties encountered when trying to show that a weight is τ-stable.

Consider $\mathsf{w}^{\#}(\gamma)$, defined in (7.33). First, the **surface term**, $e^{-\beta\|\gamma\|}$, can always be bounded using Peierls' condition:

$$\|\gamma\| = \mathscr{H}_{\Phi^0}(\omega_{\overline{\gamma}} \,|\, \eta^{\#}) + \sum_{B\subset\overline{\gamma}} \{W_B(\omega_{\overline{\gamma}}) - W_B(\eta^{\#})\} \geq (\rho - 2\|W\|)|\overline{\gamma}|.$$

Remember from Example 7.12 that, for this model, Peierls' constant can be chosen to be $\rho = 5^{-d}$. Since $\|W\| \leq |h| + |\lambda|$, from now on we will always assume that $(\lambda, h) \in U$, where

$$U \stackrel{\text{def}}{=} \left\{(\lambda, h) \in \mathbb{R}^2 : |\lambda| \leq \rho/8, |h| \leq \rho/8\right\}, \tag{7.39}$$

which gives $\rho - 2\|W\| \geq \rho/2 \stackrel{\text{def}}{=} \rho_0$, yielding

$$e^{-\beta\|\gamma\|} \leq e^{-\beta\rho_0|\overline{\gamma}|}. \tag{7.40}$$

Let us then turn to the ratio of partition functions in (7.33).

The first observation is that this ratio is always *a boundary term: there exists a constant $c > 0$ (depending on Φ) such that*

$$e^{-c\beta|\partial^{\text{ex}}\text{int}_{\#'}\gamma|} \leq \frac{\mathbf{Z}^{\#'}(\text{int}_{\#'}\gamma)}{\mathbf{Z}^{\#}(\text{int}_{\#'}\gamma)} \leq e^{+c\beta|\partial^{\text{ex}}\text{int}_{\#'}\gamma|}.$$

(To check this, the reader can use the same type of arguments that were applied to prove, in Chapter 3, that the pressure of the Ising model does not depend on the boundary condition used.) Using (7.40)*, this gives* $\mathsf{w}^{\#}(\gamma) \leq e^{-(\rho_0-c)\beta|\overline{\gamma}|}$*. Unfortunately, one can certainly not guarantee that $\rho_0 > c$. This naive argument shows that a more careful analysis is necessary to study those ratios, in order for the surface term to always be dominant.* ◇

Let us then consider the weight $\mathsf{w}^{\#}(\gamma)$, but this time expressed as in (7.35). Since the ratios of polymer partition functions in that expression induce an intricate dependence of $\mathsf{w}^{\#}(\gamma)$ on (λ, h), let us ignore this ratio for a while and assume that

$$\prod_{\#'} \frac{\Xi^{\#'}(\text{int}_{\#'}\gamma)}{\Xi^{\#}(\text{int}_{\#'}\gamma)} = 1. \tag{7.41}$$

Of course, this is a serious over-simplification, since this ratio involves in general volume terms. (Note, however, that (7.41) indeed holds when each maximal component of each $\text{int}_{\#'}\gamma$ is thin; remember (7.25).) Nevertheless, what remains of the weight after this simplification still contains volume terms, and the discussion below aims at showing how these will be handled.

Since $\mathscr{H}_{\text{int}_{\#}\gamma;\Phi}(\eta^{\#}) = \mathsf{e}^{\#}|\text{int}_{\#}\gamma|$, assuming (7.41) leave us with

$$\mathsf{w}^{\#}(\gamma) = e^{-\beta\|\gamma\|}\prod_{\#'} e^{\beta(-\mathsf{e}^{\#'}+\mathsf{e}^{\#})|\text{int}_{\#'}\gamma|} = e^{-\beta\|\gamma\|}\prod_{\#'} e^{\beta(\widehat{\psi}_0^{\#'}-\widehat{\psi}_0^{\#})|\text{int}_{\#'}\gamma|}, \tag{7.42}$$

where we have introduced, for later convenience,

$$\widehat{\psi}_0^{\#} \stackrel{\text{def}}{=} -\mathsf{e}^{\#}. \tag{7.43}$$

Therefore, to guarantee that (7.42) decays exponentially fast with $|\overline{\gamma}|$, the key issue is to verify that the **volume term**, $\prod_{\#'} e^{\beta(\widehat{\psi}_0^{\#'}-\widehat{\psi}_0^{\#})|\mathrm{int}_{\#'}\gamma|}$, is not too large to destroy the exponential decay due to the surface term.

Of course, the simplest way to guarantee this is to assume that the exponents satisfy $\widehat{\psi}_0^{\#'}-\widehat{\psi}_0^{\#}\le 0$ for each $\#'$, which occurs exactly when $(\lambda,h)\in\mathscr{U}^{\#}$ (see (7.21)), since we then have

$$\prod_{\#'} e^{\beta(\widehat{\psi}_0^{\#'}-\widehat{\psi}_0^{\#})|\mathrm{int}_{\#'}\gamma|}\le 1\,.$$

This implies that the weight of $\gamma\in\mathscr{C}^{\#}$ is $\beta\rho_0$-stable uniformly on $\mathscr{U}^{\#}$:

$$\sup_{(\lambda,h)\in\mathscr{U}^{\#}} \mathrm{w}^{\#}(\gamma)\le e^{-\beta\rho_0|\overline{\gamma}|}\,.$$

This bound is very natural, since Peierls' condition ensures that the creation of any contour represents a cost proportional to its support whenever $\eta^{\#}$ is a ground state for the pair (λ,h).

However, the construction of the phase diagram will require controlling the weights $\mathrm{w}^{\#}(\gamma)$ *in a neighborhood of the boundary of* $\mathscr{U}^{\#}$, that is, also for some values $(\lambda,h)\not\in\mathscr{U}^{\#}$, for which $\widehat{\psi}_0^{\#'}-\widehat{\psi}_0^{\#}>0$. In such a case, the volume term can be allowed to become large, but always less than the surface term. One can, for example, impose that

$$\prod_{\#'} e^{\beta(\widehat{\psi}_0^{\#'}-\widehat{\psi}_0^{\#})|\mathrm{int}_{\#'}\gamma|}\le e^{\frac{1}{2}\beta\rho_0|\overline{\gamma}|}\,. \tag{7.44}$$

To guarantee this, one will impose restrictions on (λ,h) that depend on the geometrical properties of γ, namely on the ratios $\frac{|\overline{\gamma}|}{|\mathrm{int}_{\#}\gamma|}$. To make this dependence more explicit, we will use the following classical inequality, whose proof can be found in Section B.14 (see Corollary B.80).

Lemma 7.28 (Isoperimetric inequality, $d\ge 2$)**.** *For all* $S\Subset\mathbb{Z}^d$,

$$|\partial^{\mathrm{ex}}S|\ge|S|^{\frac{d-1}{d}}\,. \tag{7.45}$$

Although we will not use them later in this precise form, consider the sets

$$\mathscr{U}_\gamma^{\#}\stackrel{\mathrm{def}}{=}\bigl\{(\lambda,h)\in U:(\widehat{\psi}_0^{\#'}-\widehat{\psi}_0^{\#})|\mathrm{int}_{\#'}\gamma|^{1/d}\le\tfrac{1}{2}\rho_0\ \text{ for each } \#'\bigr\}\,. \tag{7.46}$$

Clearly, $\mathscr{U}_\gamma^{\#}\supset\mathscr{U}^{\#}$. Taking $(\lambda,h)\in\mathscr{U}_\gamma^{\#}$, we can use the isoperimetric inequality as follows:

$$\begin{aligned}\sum_{\#'}(\widehat{\psi}_0^{\#'}-\widehat{\psi}_0^{\#})|\mathrm{int}_{\#'}\gamma| &= \sum_{\#'}(\widehat{\psi}_0^{\#'}-\widehat{\psi}_0^{\#})|\mathrm{int}_{\#'}\gamma|^{1/d}|\mathrm{int}_{\#'}\gamma|^{(d-1)/d} \\ &\le \tfrac{1}{2}\rho_0\sum_{\#'}|\partial^{\mathrm{ex}}\mathrm{int}_{\#'}\gamma| \\ &= \tfrac{1}{2}\rho_0|\partial^{\mathrm{ex}}\mathrm{int}\gamma| \\ &\le \tfrac{1}{2}\rho_0|\overline{\gamma}|\,.\end{aligned} \tag{7.47}$$

(We used the fact that the sets $\partial^{\mathrm{ex}}\mathrm{int}_{\#'}\gamma$ are pairwise disjoint subsets of $\overline{\gamma}$.) This implies (7.44) and yields $\frac{1}{2}\beta\rho_0$-stability uniformly on $\mathscr{U}_\gamma^{\#}$:

$$\sup_{(\lambda,h)\in\mathscr{U}_\gamma^{\#}} \mathrm{w}^{\#}(\gamma)\le e^{-\frac{1}{2}\beta\rho_0|\overline{\gamma}|}\,.$$

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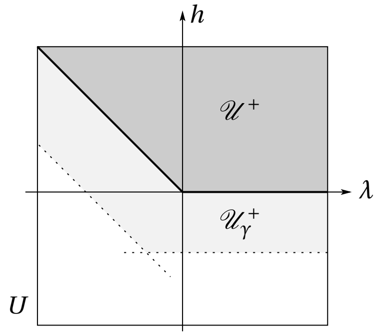


Figure 7.7: The weight of a contour $\gamma \in \mathscr{C}^+$ is $\frac{1}{2}\beta\rho_0$-stable in a region $\mathscr{U}_\gamma^+ \supset \mathscr{U}^+$, in particular in a neighborhood of the boundary of $\mathscr{U}^\#$. On the strip $\mathscr{U}_\gamma^+ \setminus \mathscr{U}^+$, which is small when the components $\mathrm{int}_-\gamma$ and $\mathrm{int}_0\gamma$ are large, (λ, h) has "the wrong sign" although $\mathrm{w}^\#(\gamma)$ remains stable.

Let's be a little bit more explicit, for example in the case $\# = +$. Since $\widehat{\psi}_0^\pm = \pm h + \lambda$ and $\widehat{\psi}_0^0 = 0$, the set $\mathscr{U}_\gamma^+$ is given by

$$\mathscr{U}_\gamma^+ = \Bigl\{(\lambda, h) \in U : h \geq -\frac{\rho_0}{4|\mathrm{int}_-\gamma|^{1/d}}\Bigr\} \cap \Bigl\{(\lambda, h) \in U : h \geq -\lambda - \frac{\rho_0}{2|\mathrm{int}_0\gamma|^{1/d}}\Bigr\},$$

and is illustrated on Figure 7.7.

At this stage, the reader might benefit from having a look at the discussion in Section 3.10.10. ◇

The above discussion provides a sketch of the method that will be used later: *controlling the balance between volume and surface terms by combining the isoperimetric inequality with relevant thermodynamic quantities depending on* (λ, h). In our simplified discussion, which occurred only at the level of ground states, the thermodynamic quantities were represented by the differences $\widehat{\psi}_0^{\#'} - \widehat{\psi}_0^\#$. In the construction of the phase diagram, the inclusion of the ratios of partition functions neglected above will represent a technical nuisance and will be treated by a proof by induction, in which $\widehat{\psi}_0^{\#'} - \widehat{\psi}_0^\#$ will be replaced by $\widehat{\psi}_n^{\#'} - \widehat{\psi}_n^\#$. The induction index n will represent the size of the largest contour present in the system (in a sense to be made precise). Starting from the ground states ($n = 0$, no contours present in the system), we will progressively add contours of increasing size. At each step n, three pressures $\widehat{\psi}_n^\#$ will be introduced, constructed using contours of size smaller or equal to n. The weights of the newly added contours of volume n will be studied in detail; one will in particular determine the regions of parameters (λ, h) for which these weights are stable.

7.4.2 Polymer models with τ-stable weights

During the induction argument below, we will use the cluster expansion to extract the surface and volume contributions to the polymer partition functions due to the quotients appearing in the weights $\mathrm{w}^\#(\gamma)$. Before pursuing, let us thus determine the conditions under which this procedure will be implemented and provide the main estimates that will be used throughout. Since cluster expansions will also be applied to auxiliary models that appear on the way, as well as to certain expressions involving derivatives with respect to λ and h, we first state a more general result, which will be applied in various situations.

Let $\mathscr{C}$ be a collection of contours, which we assume to be a subcollection of any of the families $\mathscr{C}^{\#}$, $\# \in \{+,0,-\}$. For example, $\mathscr{C}$ can be the set of all contours of type + whose interior has a size bounded by a constant. Assume that to each $\gamma \in \mathscr{C}$ corresponds a weight $\mathsf{w}(\gamma) \geq 0$, possibly different from $\mathsf{w}^{\#}(\gamma)$. We also assume that $\mathscr{C}$ and the weights $\mathsf{w}(\cdot)$ are translation invariant, in the sense that if $\gamma \in \mathscr{C}$ and if γ' is any translate of γ, then $\gamma' \in \mathscr{C}$ and $\mathsf{w}(\gamma') = \mathsf{w}(\gamma)$. We will denote the size of the support of the smallest contour of $\mathscr{C}$ by

$$\ell_0 \stackrel{\text{def}}{=} \min\{|\overline{\gamma}| : \gamma \in \mathscr{C}\}.$$

For all $\Lambda \Subset \mathbb{Z}^d$, define

$$\Xi(\Lambda) \stackrel{\text{def}}{=} \sum_{\substack{\Gamma \subset \mathscr{C}\\ \text{compatible}}} \prod_{\gamma \in \Gamma} \mathsf{w}(\gamma), \tag{7.48}$$

where the sum is over all families of pairwise compatible (in the sense of Definition 7.21) families Γ, such that $\overline{\gamma} \subset \Lambda$ and $d_\infty(\overline{\gamma}, \Lambda^{\mathrm{c}}) > 1$ for all $\gamma \in \Gamma$.

Exercise 7.13. *Show that the following limit exists:*

$$g \stackrel{\text{def}}{=} \lim_{k\to\infty} \frac{1}{|\mathsf{B}(k)|} \log \Xi(\mathsf{B}(k)). \tag{7.49}$$

Hint: *use a subadditivity argument.*

The cluster expansion for $\log \Xi(\Lambda)$, when it converges, is given by

$$\log \Xi(\Lambda) = \sum_X \Psi(X), \tag{7.50}$$

where the sum is over clusters X made of contours $\gamma \in \mathscr{C}$ such that $\overline{\gamma} \subset \Lambda$ and $d_\infty(\overline{\gamma}, \Lambda^{\mathrm{c}}) > 1$, and $\Psi(X)$ is defined as in (5.20):

$$\Psi(X) \stackrel{\text{def}}{=} \alpha(X) \prod_{\gamma \in X} \mathsf{w}(\gamma). \tag{7.51}$$

Remember that a contour can appear more than once in a cluster (in which case its weight appears more than once in the previous product), and that $\alpha(X)$ is a purely combinatorial factor.

Everywhere below, we will use the function

$$\eta(\tau, \ell) \stackrel{\text{def}}{=} 2e^{-\tau\ell/3}.$$

Theorem 7.29. *Assume that, for all $\gamma \in \mathscr{C}$, the weight $\mathsf{w}(\gamma)$ is C^1 in a parameter $s \in (a,b)$, and that, uniformly on (a,b),*

$$\mathsf{w}(\gamma) \le e^{-\tau|\overline{\gamma}|}, \qquad \Big|\frac{\mathrm{d}\mathsf{w}(\gamma)}{\mathrm{d}s}\Big| \le D|\overline{\gamma}|^{d/(d-1)} e^{-\tau|\overline{\gamma}|}, \tag{7.52}$$

where $D \ge 1$ is a constant. There exists $\tau_1 = \tau_1(D,d) < \infty$ such that the following holds. If $\tau > \tau_1$, then g defined in (7.49) *is given by the following absolutely convergent series,*

$$g = \sum_{X:\overline{X}\ni 0} \frac{1}{|\overline{X}|}\Psi(X), \tag{7.53}$$

where the sum is over clusters X made of contours $\gamma \in \mathscr{C}$ and $\overline{X} \stackrel{\text{def}}{=} \bigcup_{\gamma\in X}\overline{\gamma}$. Moreover,

$$|g| \le \eta(\tau,\ell_0) \le 1,$$

and, for all $\Lambda \Subset \mathbb{Z}^d$, g provides the volume contribution to $\log \Xi(\Lambda)$*, in the sense that*

$$\Xi(\Lambda) = \exp\big(g|\Lambda| + \Delta\big), \tag{7.54}$$

where Δ is a boundary term:

$$|\Delta| \le \eta(\tau,\ell_0)|\partial^{\mathrm{in}}\Lambda|.$$

Finally, g is also C^1 in $s \in (a,b)$, its derivative equals

$$\frac{\mathrm{d}g}{\mathrm{d}s} = \sum_{X:\overline{X}\ni 0} \frac{1}{|\overline{X}|}\frac{\mathrm{d}\Psi(X)}{\mathrm{d}s} \tag{7.55}$$

and

$$\Big|\frac{\mathrm{d}g}{\mathrm{d}s}\Big| \le D\eta(\tau,\ell_0).$$

We can express (7.54) *in the following manner*

$$\frac{1}{|\Lambda|}\log \Xi(\Lambda) = g + O\Big(\frac{|\partial^{\mathrm{in}}\Lambda|}{|\Lambda|}\Big).$$

◇

We have already seen in (7.38) that, when $\mathsf{w}(\gamma) \le e^{-\tau|\overline{\gamma}|}$, a sufficient condition for the convergence of the cluster expansion is that

$$\sum_{\gamma\in\mathscr{C}:\overline{\gamma}\ni 0} e^{-\tau|\overline{\gamma}|} e^{3^d|\overline{\gamma}|} \le 1. \tag{7.56}$$

We will actually choose τ so large that a stronger condition is satisfied, which will be needed in the proofs of Theorem 7.29 and Lemma 7.31.

Lemma 7.30. *There exists $\tau_0 < \infty$ such that, when $\tau > \tau_0$,*

$$\sum_{\gamma\in\mathscr{C}:\overline{\gamma}\ni 0} |\overline{\gamma}|^{d/(d-1)} e^{-(\tau/2-1)|\overline{\gamma}|} e^{3^d|\overline{\gamma}|} \le \eta(\tau,\ell_0) \le 1, \tag{7.57}$$

uniformly in the collection $\mathscr{C}$.

Proof. First,

$$\sum_{\substack{\gamma\in\mathscr{C}:\\ \overline{\gamma}\ni 0}} |\overline{\gamma}|^{d/(d-1)} e^{-(\tau/2-1)|\overline{\gamma}|} e^{3^d|\overline{\gamma}|} \le \sum_{k\ge \ell_0} k^{d/(d-1)} e^{-(\tau/2-1-3^d)k} \#\big\{\gamma\in\mathscr{C} : \overline{\gamma}\ni 0, |\overline{\gamma}|=k\big\}\,.$$

Once the support $\overline{\gamma}$ is fixed, the number of possible configurations $\omega_{\overline{\gamma}}$ is bounded above by $|\Omega_0|^{|\overline{\gamma}|} = 3^{|\overline{\gamma}|}$. Therefore, proceeding as in Exercise 5.3, we can show that there exists a constant $c>0$ (depending on the dimension, different from the one of Exercise 5.3) such that

$$\#\big\{\gamma\in\mathscr{C} : \overline{\gamma}\ni 0, |\overline{\gamma}|=k\big\} \le e^{ck}\,.$$

We assume that τ is so large that

$$\tau' \stackrel{\text{def}}{=} \tau/2-1-3^d-d/(d-1)-c \ge \tau/3 \tag{7.58}$$

and $e^{-\tau'}<1/2$. Then, since $k^{d/(d-1)} < e^{dk/(d-1)}$,

$$\sum_{\gamma\in\mathscr{C}:\overline{\gamma}\ni 0} |\overline{\gamma}|^{d/(d-1)} e^{-(\tau/2-1)|\overline{\gamma}|} e^{3^d|\overline{\gamma}|} \le \sum_{k\ge\ell_0} e^{-\tau' k} = \frac{e^{-\tau'\ell_0}}{1-e^{-\tau'}} \le 2e^{-\tau'\ell_0} \le \eta(\tau,\ell_0)\,.$$

For each $\#\in\{+,0,-\}$, let $\tau^\#$ be the smallest constant τ satisfying the above requirements when using $\mathscr{C}=\mathscr{C}^\#$. We can then take $\tau_0 \stackrel{\text{def}}{=} \max_\# \tau^\#$. □

Everywhere below, τ_0 will refer to the number that appeared in Lemma 7.30.

Proof of Theorem 7.29: Denote by τ_1 the smallest $\tau>\tau_0$ such that $D\eta(\tau,\ell_0)\le 1$. Since (7.57) implies (7.56), Theorem 5.4 guarantees that the series on the right-hand side of (7.50) converges absolutely. In fact, proceeding as in (5.29), using the fact that $a(\gamma) \stackrel{\text{def}}{=} |[\overline{\gamma}]| \le 3^d|\overline{\gamma}|$,

$$\sum_{X:\overline{X}\ni 0} |\Psi(X)| \le \sum_{\gamma_1\in\mathscr{C}:\overline{\gamma}_1\ni 0} \mathrm{w}(\gamma_1) e^{3^d|\overline{\gamma}_1|} \le \eta(\tau,\ell_0)\,. \tag{7.59}$$

This yields the convergence of the series for g, as well as the upper bound $|g|\le\eta(\tau,\ell_0)$. The same computations as those preceding Remark 5.9 and translation invariance give (7.54). The boundary term Δ is bounded in the same way. Since g is defined by an absolutely convergent series, we can rearrange its terms as $g=\sum_n f_n$, where

$$f_n \stackrel{\text{def}}{=} \sum_{k\ge 1} \sum_{\substack{X=\{\gamma_1,\dots,\gamma_k\}:\\ \overline{X}\ni 0\\ \sum|\gamma_i|=n}} \frac{1}{|\overline{X}|}\Psi(X)\,.$$

Since only finitely many terms contribute to f_n and since each $\Psi(X)$ is C^1, f_n is also C^1. Moreover, for a cluster $X=\{\gamma_1,\dots,\gamma_k\}$, (7.52) gives

$$\begin{aligned}\Big|\frac{\mathrm{d}\Psi(X)}{\mathrm{d}s}\Big| &= \Big|\alpha(X)\sum_{j=1}^k \frac{\mathrm{dw}(\gamma_j)}{\mathrm{d}s} \prod_{\substack{i=1\\(i\neq j)}}^k \mathrm{w}(\gamma_i)\Big| \\ &\le |\alpha(X)| \prod_{j=1}^k \big\{D|\overline{\gamma}_j|^{d/(d-1)} e^{-(\tau-1)|\overline{\gamma}_j|}\big\} = |\overline{\Psi}(X)|\,,\end{aligned}$$

where we used $\sum_{j=1}^k 1 \le \prod_{j=1}^k e^{|\overline{\gamma}_j|}$; $\overline{\Psi}(X)$ is defined as $\Psi(X)$ in (7.51), but with $\mathsf{w}(\gamma)$ replaced by

$$\overline{\mathsf{w}}(\gamma) \stackrel{\text{def}}{=} D|\overline{\gamma}|^{d/(d-1)} e^{-(\tau-1)|\overline{\gamma}|}\,.$$

Again by Lemma 7.30, the analogue of (7.56) with $\overline{\mathsf{w}}(\gamma)$ replaced by $\mathsf{w}(\gamma)$ is satisfied. Therefore,

$$\sum_n \Big|\frac{\mathrm{d}f_n}{\mathrm{d}s}\Big| \le \sum_{n\ge1}\sum_{k\ge1}\sum_{\substack{X=\{\gamma_1,\dots,\gamma_k\}:\\ \overline{X}\ni 0\\ \sum|\gamma_i|=n}} \frac{1}{|\overline{X}|}|\overline{\Psi}(X)|$$

$$\le \sum_{X:\overline{X}\ni 0} |\overline{\Psi}(X)| \le \sum_{\gamma_1\in\mathscr{C}:\overline{\gamma}_1\ni 0} \overline{\mathsf{w}}(\gamma_1) e^{3^d|\overline{\gamma}_1|} \le D\eta(\tau,\ell_0)\,.$$

Theorem B.7 thus guarantees that g is also C^1, and that its derivative is given by $\frac{\mathrm{d}g}{\mathrm{d}s} = \sum_n \frac{\mathrm{d}f_n}{\mathrm{d}s}$, which proves (7.55). □

We will also need bounds on the sums of the weights of clusters that contain at least one contour with a large support.

Lemma 7.31. *Assume that* $\mathsf{w}(\gamma) \le e^{-\tau|\overline{\gamma}|}$ *for each* $\gamma\in\mathscr{C}$ *and some* $\tau > \tau_0$. *Then, for all* $L \ge \ell_0$,

$$\sum_{\substack{X:\overline{X}\ni 0\\ |\overline{X}|\ge L}} |\Psi(X)| \le e^{-\frac12\tau L}\,. \tag{7.60}$$

Proof. Proceeding as we did at the end of Section 5.7.4,

$$1 = e^{-\tau|\overline{X}|/2} e^{\tau|\overline{X}|/2} \le e^{-\tau|\overline{X}|/2} \prod_{\gamma\in X} e^{\tau|\overline{\gamma}|/2}\,,$$

which can be inserted into

$$\sum_{\substack{X:\overline{X}\ni 0\\ |\overline{X}|\ge L}} |\Psi(X)| \le e^{-\tau L/2} \sum_{X:\overline{X}\ni 0} |\Psi(X)| \prod_{\gamma\in X} e^{\frac{\tau}{2}|\overline{\gamma}|}$$

$$= e^{-\tau L/2} \sum_{X:\overline{X}\ni 0} |\overline{\Psi}(X)| \le e^{-\tau L/2}\eta(\tau,\ell_0) \le e^{-\tau L/2}\,.$$

In the equality, we defined $\overline{\Psi}(X)$ as $\Psi(X)$ in (7.51), but with $\mathsf{w}(\gamma)$ replaced by $\overline{\mathsf{w}}(\gamma) \stackrel{\text{def}}{=} e^{-\frac{\tau}{2}|\overline{\gamma}|}$. We then used again Lemma 7.30. □

7.4.3 Truncated weights and pressures, upper bounds on partition functions

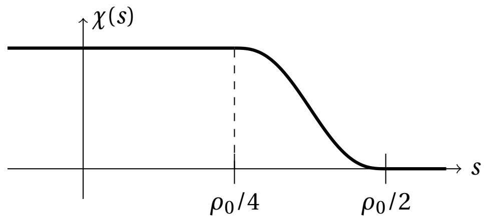
As explained above, we will construct the phase diagram by progressively adding contours on top of the ground states $\eta^{\#}$. To this end, we need to order contours according to their sizes.

Definition 7.32. *A contour* $\gamma\in\mathscr{C}^{\#}$ *is* ***of class n*** *if* $|\mathrm{int}\gamma| = n$. *The* ***collection of contours of type # and class n*** *is denoted by* $\mathscr{C}^{\#}_n$.

Clearly, a component of the interior of a contour $\gamma \in \mathscr{C}_n^\#$ can contain only contours of class strictly smaller than n.

We know that the weight of a contour with a large interior might not be stable for all values of the fields (λ, h). It will however turn out to be very useful to control the weights on the whole region $(\lambda, h) \in U$. To deal with the problem of unstable phases, we will *truncate* the weight of a contour as soon as its volume term becomes too large.

Contours of class zero contain no volume term, so they need not to be truncated. Contours of large class do however contain volume terms and we will suppress the latter as soon as they become too important. We therefore fix some choice of **cutoff function** $\chi : \mathbb{R} \to [0,1]$, satisfying the following properties: (i) $\chi(s) = 1$ if $s \le \rho_0/4$, (ii) $\chi(s) = 0$ if $s \ge \rho_0/2$, (iii) χ is C^1. Such a cutoff satisfies $\|\chi'\| \stackrel{\text{def}}{=} \sup_s |\chi'(s)| < \infty$.



We start by defining the truncated quantities associated to $n = 0$. First, the **truncated pressures** (which we already encountered before) are defined by

$$\widehat{\psi}_0^\# \stackrel{\text{def}}{=} -\mathsf{e}^\# .$$

We define

$$\forall \gamma \in \mathscr{C}_0^\#, \quad \widehat{\mathsf{w}}^\#(\gamma) \stackrel{\text{def}}{=} \mathsf{w}^\#(\gamma) = e^{-\beta\|\gamma\|} .$$

Everywhere below, we assume that $(\lambda, h) \in U$ so that we can use the bound (7.40).

Assume now that the truncated weights $\widehat{\mathsf{w}}^\#(\cdot)$ have been defined for all contours of class $\le n$. For a c-connected $\Lambda \Subset \mathbb{Z}^d$, let $\widehat{\Xi}_n^\#(\Lambda)$ denote the polymer model defined as in (7.34), but where the collections contain only contours of class $\le n$, and with $\mathsf{w}^\#(\gamma)$ replaced by $\widehat{\mathsf{w}}^\#(\gamma)$. Then, set

$$\widehat{\mathbf{Z}}_n^\#(\Lambda) \stackrel{\text{def}}{=} e^{-\beta \mathsf{e}^\# |\Lambda|} \, \widehat{\Xi}_n^\#(\Lambda) .$$

For each $\# \in \{+, 0, -\}$, we use Exercise 7.13 to define the **truncated pressure** by

$$\begin{aligned}\widehat{\psi}_n^\# = \widehat{\psi}_n^\#(\lambda, h) &\stackrel{\text{def}}{=} \lim_{k\to\infty} \frac{1}{\beta|\mathsf{B}(k)|} \log \widehat{\mathbf{Z}}_n^\#(\mathsf{B}(k)) \\ &= -\mathsf{e}^\# + \lim_{k\to\infty} \frac{1}{\beta|\mathsf{B}(k)|} \log \widehat{\Xi}_n^\#(\mathsf{B}(k)) .\end{aligned}$$

Notice that, since $\widehat{\Xi}_n^\#(\mathsf{B}(k)) \ge 1$, we have

$$\widehat{\psi}_n^\# \ge -\mathsf{e}^\# . \tag{7.61}$$

Definition 7.33. *The **truncated weight** of $\gamma \in \mathscr{C}^{\#}_{n+1}$ is defined by*

$$\widehat{\mathrm{w}}^{\#}(\gamma) \stackrel{\mathrm{def}}{=} e^{-\beta\|\gamma\|} \prod_{\#'} \Big\{ \chi\big((\widehat{\psi}^{\#'}_n - \widehat{\psi}^{\#}_n)|\mathrm{int}_{\#'}\gamma|^{1/d}\big) \frac{\mathbf{Z}^{\#'}(\mathrm{int}_{\#'}\gamma)}{\mathbf{Z}^{\#}(\mathrm{int}_{\#'}\gamma)} \Big\}.$$

Intuitively, the goal of the previous definition is to eliminate all contours that could lead to instability. The reason we do not simply use a hard constraint of the form $\mathbf{1}_{\{(\widehat{\psi}^{\#'}_n - \widehat{\psi}^{\#}_n)|\mathrm{int}_{\#'}\gamma|^{1/d} \le \rho_0/2\}}$, rather than a soft cutoff, is that the smoothness of the latter will allow us to obtain useful information on the regularity of the pressure and of the phase diagram. ⋄

Notice that since $0 \le \chi \le 1$, we have

$$\widehat{\mathrm{w}}^{\#}(\gamma) \le \mathrm{w}^{\#}(\gamma), \qquad \forall \gamma \in \mathscr{C}^{\#}. \tag{7.62}$$

Actually, unlike the true pressure ψ of the model, the truncated pressures *do* in fact depend very much on the choice of the boundary condition, that is, $\widehat{\psi}^{\#}_n \neq \widehat{\psi}^{\#'}_n$ in general. Moreover, the truncated weights and pressures depend on the specific choice of the cutoff function. Of course, as we will see later in Remark 7.37, this has no impact on the final construction of the phase diagram (but has an influence on what information on the latter can be extracted from our construction).

Other useful quantities will be important in the sequel. The first is

$$\widehat{\psi}_n \stackrel{\mathrm{def}}{=} \max_{\#} \widehat{\psi}^{\#}_n .$$

Then, the following will be handy to relate the original weights to their truncated versions:

$$a^{\#}_n \stackrel{\mathrm{def}}{=} \max_{\#'} \{\widehat{\psi}^{\#'}_n - \widehat{\psi}^{\#}_n\} = \widehat{\psi}_n - \widehat{\psi}^{\#}_n .$$

By definition, $a^{\#}_n \ge 0$ and, for all $\gamma \in \mathscr{C}^{\#}_{n+1}$,

$$a^{\#}_n |\mathrm{int}\gamma|^{1/d} \le \rho_0/4 \quad \Longrightarrow \quad \widehat{\mathrm{w}}^{\#}(\gamma) = \mathrm{w}^{\#}(\gamma). \tag{7.63}$$

The following proposition is the main technical result of this chapter. Remember that τ_1 was defined in Theorem 7.29.

Proposition 7.34. *Let*

$$\tau \stackrel{\text{def}}{=} \tfrac{1}{2}\beta\rho_0 - 6\,. \tag{7.64}$$

There exists $0 < \beta_0 < \infty$ *such that the following holds. If* $\beta > \beta_0$*, then* $\tau > \tau_1$ *and there exists an increasing sequence* $c_n \uparrow c_\infty < \infty$ *such that, for all* # *and all* $n \geq 0$*, the following statements hold.*

1. (Bounds on the truncated weights.) *For all* $k \leq n$*, the weight of each* $\gamma \in \mathscr{C}_k^\#$ *is* τ*-stable uniformly on* U*:*

$$\widehat{\mathrm{w}}^\#(\gamma) \leq e^{-\tau|\overline{\gamma}|}\,, \tag{7.65}$$

and

$$a_n^\# |\mathrm{int}\gamma|^{1/d} \leq \rho_0/8 \quad \textit{implies} \quad \widehat{\mathrm{w}}^\#(\gamma) = \mathrm{w}^\#(\gamma)\,. \tag{7.66}$$

Moreover, $\lambda \mapsto \widehat{\mathrm{w}}^\#(\gamma)$ *and* $h \mapsto \widehat{\mathrm{w}}^\#(\gamma)$ *are* C^1 *and, uniformly on* U*,*

$$\Big|\frac{\partial \widehat{\mathrm{w}}^\#(\gamma)}{\partial\lambda}\Big| \leq D|\overline{\gamma}|^{d/(d-1)} e^{-\tau|\overline{\gamma}|}\,, \qquad \Big|\frac{\partial \widehat{\mathrm{w}}^\#(\gamma)}{\partial h}\Big| \leq D|\overline{\gamma}|^{d/(d-1)} e^{-\tau|\overline{\gamma}|}\,, \tag{7.67}$$

where $D \stackrel{\text{def}}{=} 4(\beta + \|\chi'\|)$*.*

2. (Bounds on the partition functions.) *Assume that* $\Lambda \Subset \mathbb{Z}^d$ *is* c*-connected and* $|\Lambda| \leq n$*. Then*

$$\mathbf{Z}^\#(\Lambda) \leq e^{\beta\widehat{\psi}_n|\Lambda| + c_n|\partial^{\mathrm{ex}}\Lambda|}\,, \tag{7.68}$$

$$\Big|\frac{\partial \mathbf{Z}^\#(\Lambda)}{\partial\lambda}\Big| \leq \beta|\Lambda| e^{\beta\widehat{\psi}_n|\Lambda| + c_n|\partial^{\mathrm{ex}}\Lambda|}\,. \tag{7.69}$$

$$\Big|\frac{\partial \mathbf{Z}^\#(\Lambda)}{\partial h}\Big| \leq \beta|\Lambda| e^{\beta\widehat{\psi}_n|\Lambda| + c_n|\partial^{\mathrm{ex}}\Lambda|}\,, \tag{7.70}$$

uniformly in $(\lambda, h) \in U$*.*

Notice that the way the proposition is formulated allows one to obtain asymptotic bounds also in the limit $n \to \infty$.

Fixing the constants. Before turning to the proof, we fix the relevant constants. Theorem 7.29 will be used repeatedly. Remember that $\eta(\tau, \ell_0) \stackrel{\text{def}}{=} 2e^{-\tau\ell_0/3}$, where $\ell_0 \geq |\mathrm{B}(1)|$ is the size of the smallest support of a contour. We assume that β_0 satisfies $\beta_0 \geq 1$ and that it is large enough to ensure that, for all $\beta > \beta_0$, we have both $\tau > \tau_1$ and

$$D3^d\,\eta(\tau, \ell_0) \leq 1\,,$$

where $D \stackrel{\text{def}}{=} 4(\beta + \|\chi'\|)$. This will, in particular, always allow us to control the boundary terms that appear when using the cluster expansion in a region Λ:

$$|\Delta| \leq \eta(\tau, \ell_0)|\partial^{\mathrm{in}}\Lambda| \leq \eta(\tau, \ell_0)3^d|\partial^{\mathrm{ex}}\Lambda| \leq |\partial^{\mathrm{ex}}\Lambda|\,. \tag{7.71}$$

We will also assume that β_0 is large enough to guarantee that

$$\forall k \geq 1\,, \qquad 2\beta^{-1}k^{1/d}e^{-\tau k^{(d-1)/d}/2} \leq \rho_0/8\,. \tag{7.72}$$

Let $c_0 \stackrel{\text{def}}{=} 2$, and

$$c_{n+1} \stackrel{\text{def}}{=} c_n + (n+1)^{1/d} e^{-\tau n^{(d-1)/d}/2}.$$

We then have $2 \le c_n \uparrow c_\infty \stackrel{\text{def}}{=} 2 + \sum_{n\ge 1}(n+1)^{1/d} e^{-\tau n^{(d-1)/d}/2}$ and we can thus assume that β_0 is so large that $c_\infty \le 3$. Finally, we assume β_0 is such that

$$\forall a > 0, \qquad \beta^{-1}\exp\bigl(-\max\{(\rho_0/4a)^{d-1}, \ell_0\}\tau/2\bigr) \le \frac{a}{2}. \tag{7.73}$$

Proof of Proposition 7.34: Let us first prove the proposition in the case $n = 0$. When $\gamma \in \mathscr{C}_0^\#$, (7.66) is always true, $\widehat{\mathrm{w}}^\#(\gamma) = e^{-\beta\|\gamma\|}$, and (see (7.40)) $e^{-\beta\|\gamma\|} \le e^{-\beta\rho_0|\overline{\gamma}|} < e^{-\tau|\overline{\gamma}|}$ when $(\lambda, h) \in U$. Since

$$\Bigl|\frac{\partial\|\gamma\|}{\partial\lambda}\Bigr| = \Bigl|\sum_{i\in\overline{\gamma}}\{(\eta_i^\#)^2 - (\omega_{\overline{\gamma}})_i^2\}\Bigr| \le 2|\overline{\gamma}|, \tag{7.74}$$

we have

$$\Bigl|\frac{\partial\widehat{\mathrm{w}}^\#(\gamma)}{\partial\lambda}\Bigr| \le 2\beta|\overline{\gamma}|e^{-\beta\|\gamma\|} < D|\overline{\gamma}|^{d/(d-1)}e^{-\tau|\overline{\gamma}|}.$$

The bound on the derivative with respect to h is obtained in exactly the same way. Finally, (7.68)–(7.70) are trivial when $|\Lambda| = 0$, since the corresponding partition functions are all equal to 1.

We now assume that the claims of the proposition have been proved up to n, and prove that they also hold for $n+1$.

▶ *Controlling the truncated pressures $\widehat{\psi}_n^\#$.* Since the contours appearing in $\widehat{\Xi}_n^\#(\mathrm{B}(k))$ are all of class at most n and since their weights are τ-stable on U by the induction hypothesis, we can use Theorem 7.29 to express $\widehat{\psi}_n^\# = -\mathrm{e}^\# + \hat{g}_n^\#$, with $\hat{g}_n^\#$ given by the absolutely convergent series (notice that now, there appears a division by β)

$$\hat{g}_n^\# = \sum_{\substack{X\in\chi_n^\#:\\ \overline{X}\ni 0}} \frac{1}{\beta|\overline{X}|}\widehat{\Psi}^\#(X), \tag{7.75}$$

where $\chi_n^\#$ is the collection of all clusters made of contours of type # and class at most n, and where $\widehat{\Psi}^\#$ are defined as in (7.51), but with the weights $\widehat{\mathrm{w}}^\#$. Moreover, $\hat{g}_n^\#$ is C^1 in λ and h on U and

$$\Bigl|\frac{\partial\hat{g}_n^\#}{\partial\lambda}\Bigr| \le D\beta^{-1}\eta(\tau,\ell_0) \le 1. \tag{7.76}$$

The same upper bound holds for the derivative with respect to h.

▶ *Studying the truncated weights of contours of class $n+1$.* We first prove that (7.65) holds when $\gamma \in \mathscr{C}_{n+1}^\#$. Observe that $\widehat{\mathrm{w}}^\#(\gamma) = 0$ whenever there exists $\#'$ such that $(\widehat{\psi}_n^{\#'} - \widehat{\psi}_n^\#)|\mathrm{int}_{\#'}\gamma|^{1/d} > \frac{1}{2}\rho_0$. So we can assume that

$$(\widehat{\psi}_n^{\#'} - \widehat{\psi}_n^\#)|\mathrm{int}_{\#'}\gamma|^{1/d} \le \tfrac{1}{2}\rho_0 \qquad \text{for all } \#'. \tag{7.77}$$

Since $|\mathrm{int}\gamma| = n+1$, all contours contributing to the partition functions appearing in $\widehat{\mathrm{w}}^\#(\gamma)$ are of type at most n. We can thus apply the induction hypothesis to deduce that, for any $\#'$,

$$\mathbf{Z}^{\#'}(\mathrm{int}_{\#'}\gamma) \le e^{\beta\widehat{\psi}_n|\mathrm{int}_{\#'}\gamma| + c_n|\partial^{\mathrm{ex}}\mathrm{int}_{\#'}\gamma|} \le e^{\beta\widehat{\psi}_n|\mathrm{int}_{\#'}\gamma| + 3|\partial^{\mathrm{ex}}\mathrm{int}_{\#'}\gamma|}.$$

(Remember that $c_n \uparrow c_\infty \le 3$.) The truncated weight of each contour contributing to $\widehat{\mathbf{Z}}_n^{\#}(\mathrm{int}_{\#'}\gamma)$ is τ-stable by the induction hypothesis. Therefore, after using (7.62), (7.54) and (7.71),

$$\mathbf{Z}^{\#}(\mathrm{int}_{\#'}\gamma) \ge \widehat{\mathbf{Z}}_n^{\#}(\mathrm{int}_{\#'}\gamma) = e^{-\beta \mathrm{e}^{\#}|\Lambda|}\widehat{\Xi}^{\#}(\Lambda) = e^{\beta\widehat{\psi}_n^{\#}|\mathrm{int}_{\#'}\gamma|+\Delta} \ge e^{\beta\widehat{\psi}_n^{\#}|\mathrm{int}_{\#'}\gamma|-|\partial^{\mathrm{ex}}\mathrm{int}_{\#'}\gamma|}. \tag{7.78}$$

Combining these two bounds, using the isoperimetric inequality as in (7.47), we obtain

$$\frac{\mathbf{Z}^{\#'}(\mathrm{int}_{\#'}\gamma)}{\mathbf{Z}^{\#}(\mathrm{int}_{\#'}\gamma)} \le e^{\beta(\widehat{\psi}_n-\widehat{\psi}_n^{\#})|\mathrm{int}_{\#'}\gamma|+4|\partial^{\mathrm{ex}}\mathrm{int}_{\#'}\gamma|} \le e^{(\frac{1}{2}\beta\rho_0+4)|\partial^{\mathrm{ex}}\mathrm{int}_{\#'}\gamma|}. \tag{7.79}$$

Bounding the cutoff function by 1, using (7.40) and $\sum_{\#'}|\partial^{\mathrm{ex}}\mathrm{int}_{\#'}\gamma| \le |\overline{\gamma}|$, we conclude that (7.65) indeed holds for γ:

$$\widehat{\mathrm{w}}^{\#}(\gamma) \le e^{-\frac{1}{2}\beta\rho_0|\overline{\gamma}|+4|\overline{\gamma}|} < e^{-\tau|\overline{\gamma}|}. \tag{7.80}$$

Let us turn to (7.67). The derivative with respect to λ equals

$$\frac{\partial\widehat{\mathrm{w}}^{\#}(\gamma)}{\partial\lambda} = -\beta\frac{\partial\|\gamma\|}{\partial\lambda}\widehat{\mathrm{w}}^{\#}(\gamma) + e^{-\beta\|\gamma\|}\sum_{\#'}\frac{\partial}{\partial\lambda}\Big\{\chi(\cdot)\frac{\mathbf{Z}^{\#'}(\mathrm{int}_{\#'}\gamma)}{\mathbf{Z}^{\#}(\mathrm{int}_{\#'}\gamma)}\Big\}\prod_{\#''\neq\#'}\Big\{\chi(\cdot)\frac{\mathbf{Z}^{\#''}(\mathrm{int}_{\#''}\gamma)}{\mathbf{Z}^{\#}(\mathrm{int}_{\#''}\gamma)}\Big\}. \tag{7.81}$$

The only term appearing in (7.81) that we have not yet estimated is

$$\Big|\frac{\partial}{\partial\lambda}\Big\{\chi\big((\widehat{\psi}_n^{\#'}-\widehat{\psi}_n^{\#})|\mathrm{int}_{\#'}\gamma|^{1/d}\big)\frac{\mathbf{Z}^{\#'}(\mathrm{int}_{\#'}\gamma)}{\mathbf{Z}^{\#}(\mathrm{int}_{\#'}\gamma)}\Big\}\Big|.$$

Using the chain rule together with (7.76), we see that the latter is bounded above by

$$4|\mathrm{int}_{\#'}\gamma|^{1/d}\|\chi'\|\frac{\mathbf{Z}^{\#'}(\mathrm{int}_{\#'}\gamma)}{\mathbf{Z}^{\#}(\mathrm{int}_{\#'}\gamma)} + \Big|\frac{\partial}{\partial\lambda}\frac{\mathbf{Z}^{\#'}(\mathrm{int}_{\#'}\gamma)}{\mathbf{Z}^{\#}(\mathrm{int}_{\#'}\gamma)}\Big|.$$

(In the second term, the cutoff was bounded by 1.) (7.79) already leads to a bound on the first term, so we only have to consider the second one. Of course,

$$\Big|\frac{\partial}{\partial\lambda}\frac{\mathbf{Z}^{\#'}(\mathrm{int}_{\#'}\gamma)}{\mathbf{Z}^{\#}(\mathrm{int}_{\#'}\gamma)}\Big| \le \frac{|\partial\mathbf{Z}^{\#'}(\mathrm{int}_{\#'}\gamma)/\partial\lambda|}{\mathbf{Z}^{\#}(\mathrm{int}_{\#'}\gamma)} + \frac{\mathbf{Z}^{\#'}(\mathrm{int}_{\#'}\gamma)}{\mathbf{Z}^{\#}(\mathrm{int}_{\#'}\gamma)}\frac{|\partial\mathbf{Z}^{\#}(\mathrm{int}_{\#'}\gamma)/\partial\lambda|}{\mathbf{Z}^{\#}(\mathrm{int}_{\#'}\gamma)}. \tag{7.82}$$

Observe that

$$\begin{aligned}\Big|\frac{\partial\mathbf{Z}^{\#}(\mathrm{int}_{\#'}\gamma)}{\partial\lambda}\Big| &= \Big|\sum_{\omega\in\Omega_{\mathrm{int}_{\#'}\gamma}} e^{-\beta\mathscr{H}_{\mathrm{int}_{\#'}\gamma;\Phi}(\omega)}\frac{\partial(-\beta\mathscr{H}_{\mathrm{int}_{\#'}\gamma;\Phi}(\omega))}{\partial\lambda}\Big|\\ &\le \beta|\mathrm{int}_{\#'}\gamma|\mathbf{Z}^{\#}(\mathrm{int}_{\#'}\gamma).\end{aligned} \tag{7.83}$$

This bounds the last ratio in (7.82). The remaining ratios can be estimated using the induction hypothesis (7.69), and (7.79), yielding

$$\Big|\frac{\partial}{\partial\lambda}\frac{\mathbf{Z}^{\#'}(\mathrm{int}_{\#'}\gamma)}{\mathbf{Z}^{\#}(\mathrm{int}_{\#'}\gamma)}\Big| \le 2\beta|\mathrm{int}_{\#'}\gamma|e^{(\frac{1}{2}\beta\rho_0+4)|\partial^{\mathrm{ex}}\mathrm{int}_{\#'}\gamma|}. \tag{7.84}$$

Using (7.40), (7.74), (7.79), (7.80) and (7.84) in (7.81) leads to

$$\Big|\frac{\partial\widehat{\mathrm{w}}^{\#}(\gamma)}{\partial\lambda}\Big| \le \big\{2\beta|\overline{\gamma}| + (2\beta+4\|\chi'\|)|\mathrm{int}\gamma|\big\}e^{-\tau|\overline{\gamma}|}.$$

(7.67) then follows after an application of the isoperimetric inequality. Once again, the derivative with respect to h is treated in the same way.

► *Estimating the differences* $|\widehat{\psi}^\#_{n+1} - \widehat{\psi}^\#_k|$, $k \le n$. Notice that $|\widehat{\psi}^\#_{n+1} - \widehat{\psi}^\#_k| = |\hat{g}^\#_{n+1} - \hat{g}^\#_k|$. By what was done above, all truncated weights of contours of class $\le n+1$ are τ-stable. We can therefore consider the expansions for each of the functions $\hat{g}^\#_k$, $k \le n+1$, as in (7.75). Then, observe that the clusters $X \in \chi^\#_{n+1} \setminus \chi^\#_k$ that contribute to $\hat{g}^\#_{n+1} - \hat{g}^\#_k$ contain at least one contour γ_* with $|\text{int}\gamma_*| > k$. By the isoperimetric inequality, $|\overline{\gamma}_*| \ge k^{(d-1)/d}$. By Lemma 7.31,

$$\sum_{\substack{X: \overline{X} \ni 0 \\ |\overline{X}| \ge k^{(d-1)/d}}} |\widehat{\Psi}^\#(X)| \le e^{-\frac{1}{2}\tau k^{(d-1)/d}}.$$

As a consequence,

$$|\widehat{\psi}^\#_{n+1} - \widehat{\psi}^\#_k| \le \beta^{-1} e^{-\frac{1}{2}\tau k^{(d-1)/d}}, \qquad |\widehat{\psi}_{n+1} - \widehat{\psi}_k| \le \beta^{-1} e^{-\frac{1}{2}\tau k^{(d-1)/d}}. \tag{7.85}$$

► *Showing that n, in* (7.66)*, can be replaced byn* $+1$. For $\gamma \in \mathscr{C}^\#_{n+1}$, (7.66) holds by (7.63). Let us fix $\gamma \in \mathscr{C}^\#_k$, $k \le n$, and write

$$\begin{aligned} a^\#_k |\text{int}\gamma|^{1/d} &= a^\#_{n+1} |\text{int}\gamma|^{1/d} + (a^\#_k - a^\#_{n+1}) |\text{int}\gamma|^{1/d} \\ &\le a^\#_{n+1} |\text{int}\gamma|^{1/d} + 2\beta^{-1} k^{1/d} e^{-\tau k^{(d-1)/d}/2} \\ &\le a^\#_{n+1} |\text{int}\gamma|^{1/d} + \rho_0/8. \end{aligned}$$

(We used (7.72) in the last inequality.) Therefore, $a^\#_{n+1}|\text{int}\gamma|^{1/d} \le \rho_0/8$ implies that $a^\#_k |\text{int}\gamma|^{1/d} \le \rho_0/4$. According to how the cutoffs were defined, this implies $\widehat{\mathrm{w}}^\#(\gamma) = \mathrm{w}^\#(\gamma)$.

We now move on to the most delicate part of the proof:

► *Showing that* (7.68) *holds if* $|\Lambda| = n+1$. Let $\Lambda \Subset \mathbb{Z}^d$ be an arbitrary c-connected set satisfying $|\Lambda| = n+1$, fix $(\lambda, h) \in U$ and consider $\mathbf{Z}^\#(\Lambda) = e^{-\beta \mathrm{e}^\# |\Lambda|} \Xi^\#(\Lambda)$. Let $\gamma \in \mathscr{C}^\#$ be any contour appearing in the contour representation of $\Xi^\#(\Lambda)$ (therefore necessarily of class at most n). If

$$a^\#_n |\text{int}\gamma|^{1/d} \le \rho_0/4,$$

we say that γ is **stable**; otherwise, we say thatγ is **unstable**. By definition, a stable contour satisfies $\mathrm{w}^\#(\gamma) = \widehat{\mathrm{w}}^\#(\gamma)$.

Whether a contour is stable or not depends on the point $(\lambda, h) \in U$ we are considering. Note that when $a^\#_n = 0$, all contours appearing in $\mathbf{Z}^\#(\Lambda)$ are stable; in such a case, we can use Theorem 7.29 to conclude that

$$\mathbf{Z}^\#(\Lambda) = \widehat{\mathbf{Z}}^\#(\Lambda) = e^{-\beta \mathrm{e}^\# |\Lambda|} \Xi^\#(\Lambda) = e^{\beta \widehat{\psi}_n |\Lambda| + \Delta} \le e^{\beta \widehat{\psi}_n |\Lambda| + |\partial^{\text{ex}} \Lambda|} \le e^{\beta \widehat{\psi}_n |\Lambda| + c_n |\partial^{\text{ex}} \Lambda|}.$$

We can thus assume that $a^\#_n > 0$. Note that the possible presence of unstable contours prevents us now from using the representation $\mathbf{Z}^\#(\Lambda) = e^{-\beta \mathrm{e}^\#} \Xi^\#(\Lambda)$ to analyze $\mathbf{Z}^\#(\Lambda)$.

Let us fix the set of external *unstable* contours. Once the latter are fixed, we can resum over the configurations on their exterior Λ^{ext}, with the restriction of allowing only stable contours. Observe that being stable is hereditary: if γ is stable, then any contour contained in its interior is also stable, so that we are guaranteed that none of these contours will surround one of the fixed unstable contours.

Proceeding similarly to what we did in (7.29), this first step gives

$$\mathbf{Z}^{\#}(\Lambda) = \sum_{\substack{\Gamma' \\ \text{compatible} \\ \text{external} \\ \text{unstable}}} \mathbf{Z}^{\#}_{\text{stable}}(\Lambda^{\text{ext}}) \prod_{\gamma' \in \Gamma'} \Big\{ \exp\big(-\beta \sum_{B \subset \overline{\gamma}'} \Phi_B(\omega_{\overline{\gamma}'})\big) \prod_{\#'} \mathbf{Z}^{\#'}(\text{int}_{\#'}\gamma') \Big\},$$

where $\mathbf{Z}^{\#}_{\text{stable}}(\Lambda^{\text{ext}})$ denotes the partition function restricted to configurations in which all contours are stable. Since $\mathrm{w}^{\#}(\gamma) = \widehat{\mathrm{w}}^{\#}(\gamma)$ when γ is stable and since the truncated weights are τ-stable, we can use a convergent cluster expansion to study $\mathbf{Z}^{\#}_{\text{stable}}(\Lambda^{\text{ext}})$:

$$\mathbf{Z}^{\#}_{\text{stable}}(\Lambda^{\text{ext}}) = e^{-\beta \mathrm{e}^{\#}|\Lambda^{\text{ext}}|} \Xi^{\#}_{\text{stable}}(\Lambda^{\text{ext}}) \le e^{-\beta(\mathrm{e}^{\#} - \hat{g}^{\#}_{n,\text{stable}})|\Lambda^{\text{ext}}|} e^{|\partial^{\text{ex}}\Lambda^{\text{ext}}|}.$$

For the partition functions in the interior of unstable contours, we apply the induction hypothesis (7.68):

$$\prod_{\#'} \mathbf{Z}^{\#'}(\text{int}_{\#'}\gamma') \le \prod_{\#'} e^{\beta \widehat{\psi}_n |\text{int}_{\#'}\gamma'| + c_n |\partial^{\text{ex}} \text{int}_{\#'}\gamma'|} \le e^{\beta \widehat{\psi}_n |\text{int}\gamma'|} e^{3|\overline{\gamma}'|}.$$

Using $|\partial^{\text{ex}}\Lambda^{\text{ext}}| \le |\partial^{\text{ex}}\Lambda| + \sum_{\gamma' \in \Gamma'} |\overline{\gamma}'|$ and extracting $e^{\beta \widehat{\psi}_n |\Lambda|}$ from the sum, we obtain

$$\mathbf{Z}^{\#}(\Lambda) \le e^{\beta \widehat{\psi}_n |\Lambda|} e^{|\partial^{\text{ex}}\Lambda|} \times \sum_{\substack{\Gamma' \\ \text{compatible} \\ \text{external} \\ \text{unstable}}} e^{-\beta(\widehat{\psi}_n + \mathrm{e}^{\#} - \hat{g}^{\#}_{n,\text{stable}})|\Lambda^{\text{ext}}|} \prod_{\gamma' \in \Gamma'} \exp\big(-\beta \sum_{B \subset \overline{\gamma}'} \Phi_B(\omega_{\overline{\gamma}'})\big) e^{(4 - \beta \widehat{\psi}_n)|\overline{\gamma}'|}.$$

Observe now that $\widehat{\psi}_n |\overline{\gamma}| \ge \widehat{\psi}^{\#}_n |\overline{\gamma}| \ge -\mathrm{e}^{\#}|\overline{\gamma}| = -\sum_{B \subset \overline{\gamma}} \Phi_B(\eta^{\#})$ (indeed, remember (7.61), and observe that all pairwise interactions, in a ground state, are zero). Defining $\widehat{\psi}^{\#}_{n,\text{stable}} \stackrel{\text{def}}{=} -\mathrm{e}^{\#} + \hat{g}^{\#}_{n,\text{stable}}$,

$$\mathbf{Z}^{\#}(\Lambda) \le e^{\beta \widehat{\psi}_n |\Lambda|} e^{|\partial^{\text{ex}}\Lambda|} \sum_{\substack{\Gamma' \\ \text{compatible} \\ \text{external} \\ \text{unstable}}} e^{-\beta(\widehat{\psi}_n - \widehat{\psi}^{\#}_{n,\text{stable}})|\Lambda^{\text{ext}}|} \prod_{\gamma' \in \Gamma'} e^{-\beta \|\gamma'\|} e^{4|\overline{\gamma}'|}. \tag{7.86}$$

We will show that this last sum is bounded by $e^{|\partial^{\text{ex}}\Lambda|}$, which will allow to conclude that, indeed,

$$\mathbf{Z}^{\#}(\Lambda) \le e^{\beta \widehat{\psi}_n |\Lambda|} e^{2|\partial^{\text{ex}}\Lambda|} \le e^{\beta \widehat{\psi}_n |\Lambda| + c_n |\partial^{\text{ex}}\Lambda|}. \tag{7.87}$$

In order to do that, we will prove that $\widehat{\psi}_n - \widehat{\psi}^{\#}_{n,\text{stable}}$ is positive and sufficiently large to strongly penalize families Γ' for which $|\Lambda^{\text{ext}}|$ is large. First, let us write

$$\widehat{\psi}_n - \widehat{\psi}^{\#}_{n,\text{stable}} = a^{\#}_n + (\hat{g}^{\#}_n - \hat{g}^{\#}_{n,\text{stable}}).$$

The clusters that contribute to $\hat{g}^{\#}_n - \hat{g}^{\#}_{n,\text{stable}}$ necessarily contain at least one unstable contour. Therefore, since an unstable contour γ satisfies

$$|\overline{\gamma}| \ge |\text{int}\gamma|^{(d-1)/d} \ge \Big(\frac{\rho_0}{4a^{\#}_n}\Big)^{d-1},$$

we have

$$|\hat{g}_n^\# - \hat{g}_{n,\mathrm{stable}}^\#| \le \beta^{-1}\exp\bigl(-\max\{(\rho_0/4a_n^\#)^{d-1},\ell_0\}\tau/2\bigr) \le \tfrac12 a_n^\#,$$

where we used (7.73). We conclude that $\widehat{\psi}_n - \widehat{\psi}^\#_{n,\mathrm{stable}} \ge a_n^\#/2$. Then, let us define new weights as follows: for each $\gamma \in \mathscr{C}^\#$, set

$$\mathsf{w}_*^\#(\gamma) \stackrel{\mathrm{def}}{=} \begin{cases} e^{-(\beta\rho_0-5)|\overline{\gamma}|} & \text{if } \gamma \text{ is unstable,}\\ 0 & \text{otherwise.}\end{cases}$$

We denote by $\Xi_*^\#(\cdot)$ the associated polymer partition function, and let

$$\hat{g}_*^\# \stackrel{\mathrm{def}}{=} \lim_{n\to\infty} \frac{1}{\beta|\mathsf{B}(n)|}\log \Xi_*^\#(\mathsf{B}(n)).$$

Since $\beta\rho_0 - 5 \ge \tau$, $\hat{g}_*^\#$ can be controlled by a convergent cluster expansion. Once again, the clusters that contribute to $\hat{g}_*^\#$ contain only unstable contours, and therefore, using again (7.73),

$$|\hat{g}_*^\#| \le \beta^{-1}\exp\bigl(-\max\{(\rho_0/4a_n^\#)^{d-1},\ell_0\}\tau/2\bigr) \le \tfrac12 a_n^\#. \tag{7.88}$$

One can thus guarantee that

$$\widehat{\psi}_n - \widehat{\psi}^\#_{n,\mathrm{stable}} \ge \hat{g}_*^\#. \tag{7.89}$$

We can now use this to show that the sum in (7.86) is bounded above by

$$\begin{aligned}
\sum_{\substack{\Gamma'\\ \text{compatible}\\ \text{external}\\ \text{unstable}}} e^{-\beta\hat{g}_*^\#|\Lambda^{\mathrm{ext}}|}\prod_{\gamma'\in\Gamma'} e^{-(\beta\rho_0-4)|\overline{\gamma}'|} &\le e^{-\beta\hat{g}_*^\#|\Lambda|}\sum_{\substack{\Gamma'\\ \text{compatible}\\ \text{external}\\ \text{unstable}}}\prod_{\gamma'\in\Gamma'} e^{-(\beta\rho_0-5)|\overline{\gamma}'|}e^{\beta\hat{g}_*^\#|\mathrm{int}\gamma'|}\\
&\le e^{-\beta\hat{g}_*^\#|\Lambda|}\sum_{\substack{\Gamma'\\ \text{compatible}\\ \text{external}\\ \text{unstable}}}\prod_{\gamma'\in\Gamma'} e^{-(\beta\rho_0-6)|\overline{\gamma}'|}\Xi_*^\#(\mathrm{int}\gamma')\\
&= e^{-\beta\hat{g}_*^\#|\Lambda|}\Xi_*^\#(\Lambda)\\
&\le e^{|\partial^{\mathrm{ex}}\Lambda|}.
\end{aligned}$$

In the first inequality, we used $|\hat{g}_*^\#| \le 1$, which follows from the first inequality in (7.88); in the second, we used again Theorem 7.29: $\Xi_*^\#(\mathrm{int}\gamma) \ge e^{\beta\hat{g}_*^\#|\mathrm{int}\gamma|-|\overline{\gamma}|}$. This proves the earlier claim.

► *Showing that* (7.69) *and* (7.70) *hold for* $|\Lambda| = n+1$. Proceeding as in (7.83), we see that

$$\Bigl|\frac{\partial \mathbf{Z}^\#(\Lambda)}{\partial\lambda}\Bigr| \le \beta|\Lambda|\mathbf{Z}^\#(\Lambda), \tag{7.90}$$

and, therefore, (7.69) follows from (7.87). The same argument yields (7.70)

► *Showing that* n*, in the right-hand side of* (7.68)–(7.70)*, can be replaced by* $n+1$. Using (7.85) and the isoperimetric inequality we get, for all $|\Lambda| \le n+1$,

$$\begin{aligned}
\beta\widehat{\psi}_n|\Lambda| + c_n|\partial^{\mathrm{ex}}\Lambda| &= \beta\widehat{\psi}_{n+1}|\Lambda| + c_n|\partial^{\mathrm{ex}}\Lambda| + \beta(\widehat{\psi}_n - \widehat{\psi}_{n+1})|\Lambda|\\
&\le \beta\widehat{\psi}_{n+1}|\Lambda| + (c_n + (n+1)^{1/d}e^{-\frac12\tau n^{(d-1)/d}})|\partial^{\mathrm{ex}}\Lambda|\\
&= \beta\widehat{\psi}_{n+1}|\Lambda| + c_{n+1}|\partial^{\mathrm{ex}}\Lambda|.
\end{aligned}$$ □

7.4.4 Construction of the phase diagram

Let us now exploit the consequences of Proposition 7.34. We assume throughout this section that $\beta > \beta_0$. Since it appears at several places, we define

$$\epsilon = \epsilon(\beta) \stackrel{\text{def}}{=} D3^d \eta(\tau, \ell_0) .$$

When needed, β can be taken larger to make ϵ smaller. We will assume, for instance, that

$$\epsilon < \rho/32 .$$

Proposition 7.34 dealt with the truncated pressures $\widehat{\psi}_n^{\#}$, to which only contours with an interior of size at most n contributed. Let us first see how the limit $n \to \infty$ restores the full model.

It follows from (7.85) that $\hat{g}_n^{\#}$ is a Cauchy sequence, which guarantees the existence of

$$\hat{g}^{\#} \stackrel{\text{def}}{=} \lim_{n\to\infty} \hat{g}_n^{\#} .$$

Moreover, $\hat{g}^{\#}$ can be expressed as the convergent series

$$\hat{g}^{\#} = \sum_{\substack{X \in \chi^{\#}: \\ \overline{X} \ni 0}} \frac{1}{\beta|\overline{X}|} \widehat{\Psi}^{\#}(X) , \tag{7.91}$$

where $\chi^{\#}$ is the collection of all clusters made of contours of type # using the weights $\widehat{\mathrm{w}}^{\#}$. Namely, the difference between this series and the one in (7.75) is an infinite sum over clusters X such that (i) their support contains 0 and (ii) they contain at least one contour of class larger than n. By Lemma 7.31,

$$|\hat{g}^{\#} - \hat{g}_n^{\#}| \leq \beta^{-1} e^{-\frac{1}{2}\tau n^{(d-1)/d}} . \tag{7.92}$$

We can thus also define

$$\widehat{\psi}^{\#} \stackrel{\text{def}}{=} \lim_{n\to\infty} \widehat{\psi}_n^{\#} .$$

The series in (7.91) can be bounded as usual: $|\hat{g}^{\#}| \leq \epsilon$. This shows that $\widehat{\psi}^{\#}$ is a small perturbation of minus the energy density of the ground state $\eta^{\#}$:

$$\left|\widehat{\psi}^{\#} - (-\mathrm{e}^{\#})\right| \leq \epsilon ,$$

In order to compare the original weights and their truncated versions, we define

$$a^{\#} \stackrel{\text{def}}{=} \lim_{n\to\infty} a_n^{\#} = \widehat{\psi} - \widehat{\psi}^{\#} ,$$

where

$$\widehat{\psi} \stackrel{\text{def}}{=} \max_{\#} \widehat{\psi}^{\#} .$$

Letting $n \to \infty$ in (7.66) implies that

$$\forall \gamma \in \mathscr{C}^{\#}, \qquad a^{\#}|\mathrm{int}\gamma|^{1/d} \leq \rho_0/8 \quad \text{implies} \quad \widehat{\mathrm{w}}^{\#}(\gamma) = \mathrm{w}^{\#}(\gamma) . \tag{7.93}$$

Using this, we can define regions of parameters on which *all contours of type # will coincide with their truncated versions* (we indicate the dependence on β, to distinguish these sets from those defined in (7.21), which were associated to the energy densities of the ground states):

$$\mathscr{U}_\beta^{\#} \stackrel{\text{def}}{=} \left\{(\lambda, h) \in U : a^{\#}(\lambda, h) = 0\right\} = \left\{(\lambda, h) \in U : \widehat{\psi}^{\#}(\lambda, h) = \max_{\#'} \widehat{\psi}^{\#'}(\lambda, h)\right\} .$$

Observe that at a given point $(\lambda, h) \in U$, there is always at least one # for which $a^{\#}(\lambda, h) = 0$. This means that the regions $\mathscr{U}^{\#}$ cover U. By (7.93), we obtain the following

Theorem 7.35. *There exists* $0 < \beta_0 < \infty$ *such that the following holds for all* $\beta > \beta_0$*:*

$$\forall \gamma \in \mathscr{C}^{\#}, \qquad (\lambda, h) \in \mathscr{U}^{\#}_{\beta} \quad \textit{implies} \quad \widehat{\mathrm{w}}^{\#}(\gamma) = \mathrm{w}^{\#}(\gamma).$$

In particular, when $(\lambda, h) \in \mathscr{U}^{\#}_{\beta}$*, the* true *pressure of the model equals*

$$\begin{aligned}\psi(\lambda, h) &= \lim_{n\to\infty} \frac{1}{\beta|\mathrm{B}(n)|} \log \mathbf{Z}^{\#}(\mathrm{B}(n)) \\ &= \lim_{n\to\infty} \frac{1}{\beta|\mathrm{B}(n)|} \log \widehat{\mathbf{Z}}^{\#}(\mathrm{B}(n)) = \widehat{\psi}^{\#}(\lambda, h).\end{aligned}$$

In other words,

$$\psi(\lambda, h) = \begin{cases} \widehat{\psi}^{+}(\lambda, h) & \text{if } (\lambda, h) \in \mathscr{U}^{+}_{\beta}, \\ \widehat{\psi}^{0}(\lambda, h) & \text{if } (\lambda, h) \in \mathscr{U}^{0}_{\beta}, \\ \widehat{\psi}^{-}(\lambda, h) & \text{if } (\lambda, h) \in \mathscr{U}^{-}_{\beta}. \end{cases} \tag{7.94}$$

In particular, we can extract properties of the (true) pressure by studying the truncated pressures and determining the regions $\mathscr{U}^{\#}_{\beta}$.

Up to now, even though we restricted our discussion to the Blume–Capel model for pedagogical reasons, the specific properties of this model were not used in any important way. In order to obtain more precise information about this model however, it will be useful to exploit these properties from now on. For example, the $+ \leftrightarrow -$ symmetry provides us immediately with useful information about the truncated pressures.

Exercise 7.14. *Check that the* $+ \leftrightarrow -$ *symmetry implies* $\widehat{\psi}^{+}(\lambda, -h) = \widehat{\psi}^{-}(\lambda, h)$.

Let us write the truncated pressures more explicitly, using the expressions for the ground state energy densities $\mathrm{e}^{\#}$ given in (7.20):

$$\widehat{\psi}^{\pm}(\lambda, h) = \pm h + \lambda + \hat{g}^{\pm}(\lambda, h), \qquad \widehat{\psi}^{0}(\lambda, h) = \hat{g}^{0}(\lambda, h).$$

Since the weights $\widehat{\mathrm{w}}^{\#}(\gamma)$ are C^1, Theorem 7.29 guarantees again that $\hat{g}^{\#}$ is C^1 on U and that, uniformly on U,

$$\Big|\frac{\partial \hat{g}^{\#}}{\partial \lambda}\Big| \le \epsilon, \qquad \Big|\frac{\partial \hat{g}^{\#}}{\partial h}\Big| \le \epsilon. \tag{7.95}$$

Therefore,

$$\Big|\frac{\partial \widehat{\psi}^{\pm}}{\partial h} \mp 1\Big| \le \epsilon, \qquad \Big|\frac{\partial \widehat{\psi}^{\pm}}{\partial \lambda} - 1\Big| \le \epsilon, \qquad \Big|\frac{\partial \widehat{\psi}^{0}}{\partial h}\Big| \le \epsilon, \qquad \Big|\frac{\partial \widehat{\psi}^{0}}{\partial \lambda}\Big| \le \epsilon.$$

The regions $\mathscr{U}^{\#}_{\beta}$. Let us start with $\mathscr{U}^{+}_{\beta}$, which can be written as

$$\mathscr{U}^{+}_{\beta} = \big\{(\lambda, h) \in U : \widehat{\psi}^{+} \ge \widehat{\psi}^{-}\big\} \cap \big\{(\lambda, h) \in U : \widehat{\psi}^{+} \ge \widehat{\psi}^{0}\big\}.$$

Since the truncated pressures are continuous, the boundary of the set $\{\widehat{\psi}^+ \geq \widehat{\psi}^-\}$ is given by $\{\widehat{\psi}^+ = \widehat{\psi}^-\}$. By the $+ \leftrightarrow -$ symmetry (Exercise 7.14),

$$\widehat{\psi}^+(\lambda, 0) = \widehat{\psi}^-(\lambda, 0), \quad \forall \lambda.$$

Since $\frac{\partial \widehat{\psi}^+}{\partial h} > \frac{\partial \widehat{\psi}^-}{\partial h}$, uniformly on U, this shows that $\{\widehat{\psi}^+ = \widehat{\psi}^-\} = \{h = 0\}$ and that $\{\widehat{\psi}^+ \geq \widehat{\psi}^-\} = \{h \geq 0\}$. In the same way, the boundary of $\{\widehat{\psi}^+ \geq \widehat{\psi}^0\}$ equals $\{\widehat{\psi}^+ = \widehat{\psi}^0\}$. Since there is no symmetry between + and 0, we will fix λ, and search for the value of h such that $\widehat{\psi}^+(\lambda, h) = \widehat{\psi}^0(\lambda, h)$, which can also be written as

$$G(\lambda, h) \stackrel{\text{def}}{=} h + \lambda + \hat{g}^+(\lambda, h) - \hat{g}^0(\lambda, h) = 0. \tag{7.96}$$

To guarantee that (7.96) has solutions in U, we restrict our attention to a slightly smaller region. Remember that U is defined by $|\lambda|, |h| < \rho/8$. Let $\delta > 0$ be any number satisfying $2\epsilon < \delta < \rho/16$. and set

$$\check{U} \stackrel{\text{def}}{=} \left\{(\lambda, h) : |\lambda| < \rho/8 - \delta, |h| < \rho/8\right\}.$$

Take then $\lambda \in (-\rho/8 + \delta, \rho/8 - \delta)$ and define $h_\pm \stackrel{\text{def}}{=} -\lambda \pm \delta$. We have $(\lambda, h_\pm) \in U$ and $G(\lambda, h_-) < 0 < G(\lambda, h_+)$. Since $\frac{\partial G}{\partial h} > 0$ uniformly on U, this implies that there exists a unique $h = h(\lambda) \in (h_-, h_+)$, such that $G(\lambda, h) = 0$. The implicit function theorem guarantees that $\lambda \mapsto h(\lambda)$ is actually C^1, and differentiating $G(\lambda, h(\lambda)) = 0$ with respect to λ leads to $|h'(\lambda) + 1| \leq 2\epsilon$.

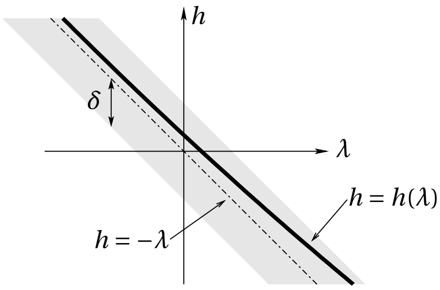


Figure 7.8: The construction of $\{\widehat{\psi}^+ = \widehat{\psi}^0\}$, which can be parametrized by a smooth map $\lambda \mapsto h(\lambda)$, whose graph lies in the strip of width 2δ around $h = -\lambda$.

One then has $\mathscr{U}_\beta^+ = \left\{(\lambda, h) \in \check{U} : h \geq \max\{h(\lambda), 0\}\right\}$. By symmetry,

$$\mathscr{U}_\beta^- = \left\{(\lambda, h) \in \check{U} : (\lambda, -h) \in \mathscr{U}_\beta^+\right\},$$

and $\{\widehat{\psi}^- = \widehat{\psi}^0\}$ can be parametrized by $\lambda \mapsto -h(\lambda)$. Finally, $\mathscr{U}_\beta^0$ is the closure of $\check{U} \setminus (\mathscr{U}_\beta^+ \cup \mathscr{U}_\beta^-)$. The regions $\mathscr{U}_\beta^\#$ are separated by **coexistence lines**,

$$\mathscr{L}_\beta^{\#\#'} \stackrel{\text{def}}{=} \mathscr{U}_\beta^\# \cap \mathscr{U}_\beta^{\#'} \qquad \# \neq \#'.$$

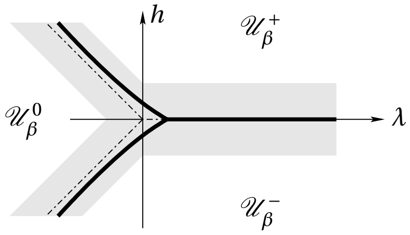


Figure 7.9: The phase diagram of the Blume–Capel model, which lies in a neighborhood of size δ of the zero-temperature diagram (dashed lines). Actually, by the $+/-$ symmetry of the model, we know that the line separating $\mathscr{U}_\beta^+$ and $\mathscr{U}_\beta^-$ lies *exactly* on the line $\{h = 0\}$.

7.4.5 Results for the pressure

We summarize the results obtained so far about the pressure in the following theorem (see also the qualitative picture on Figure 7.3).

Theorem 7.36 (The pressure of the Blume–Capel model at low temperature)**.** *Let β_0 be as in Proposition 7.34. For all $\beta > \beta_0$,*

$$\psi(\lambda, h) = \max_{\#} \widehat{\psi}^{\#}(\lambda, h) = \begin{cases} \widehat{\psi}^{+}(\lambda, h) & \text{if } (\lambda, h) \in \mathscr{U}_\beta^+, \\ \widehat{\psi}^{0}(\lambda, h) & \text{if } (\lambda, h) \in \mathscr{U}_\beta^0, \\ \widehat{\psi}^{-}(\lambda, h) & \text{if } (\lambda, h) \in \mathscr{U}_\beta^-. \end{cases} \tag{7.97}$$

As a consequence,

1. *The pressure $\psi(\lambda, h)$ is C^1 in λ and h, everywhere in the interior of each region $\mathscr{U}_\beta^\#$, $\# \in \{+, 0, -\}$.*
2. ***First-order phase transitions** occur across each of the coexistence lines, in the sense that*
$$\frac{\partial \psi}{\partial \lambda^+} > \frac{\partial \psi}{\partial \lambda^-}, \qquad \text{at each } (\lambda, h) \in \mathscr{L}_\beta^{\pm 0}.$$
and, for all $\# \neq \#'$,
$$\frac{\partial \psi}{\partial h^+} > \frac{\partial \psi}{\partial h^-} \qquad \text{at each } (\lambda, h) \in \mathscr{L}_\beta^{\#\#'}.$$

Remark 7.37. Remember that the construction of the truncated pressures depends on the choice of the cutoff $\chi(\cdot)$ used in the definition of the truncated weights. The latter choice of course only affects the truncated pressures. It has however an impact on what we could extract from the analysis above. Namely, the assumption that the cutoff was C^1 yields, ultimately, the corresponding regularity of the pressure in the interior of the regions $\mathscr{U}_\beta^\#$, as well as the regularity of the boundary of these regions. Choosing a cutoff with higher regularity would yield a corresponding enhancement of these properties (but would require a control of higher-order derivatives of the truncated pressures in Proposition 7.34). ⋄

Remark 7.38. The fact that the pressure coincides with the maximal truncated pressure (see (7.97)) means that the truncated pressures provide natural continuations of the pressure through the coexistence lines. A similar conclusion had been drawn for the Curie–Weiss model (see Figure 3.19 of page 158). Nevertheless, the continuations of $h \mapsto \psi_\beta^{\mathrm{CW}}(h)$ through the transition point were *analytic*, while those obtained here are only C^1. In particular, there are infinitely-many possibilities to make such a continuation, in contrast with the analytic case. For example, each choice of a cut-off function yields an a priori different C^1 continuation. Nevertheless, *analytic* continuations through the coexistence lines do not exist, in the general, in the framework of PST. This will be discussed in Section 7.6.6, ⋄

Remark 7.39. In this chapter, we started with the Blume–Capel model at parameters $h = \lambda = 0$ and considered perturbations around this point, constructing the phase diagram in its vicinity. In the same way, we could have started with $\lambda = \lambda_0$ and $h = h_0$, and constructed the phase diagram around this point. This allows in particular the description of coexistence lines outside the domain U. ⋄

7.4.6 The Gibbs measures at low temperature

So far, the phase diagram was constructed by studying partition functions and truncated pressures. In this section, we consider the consequences of the previous study at low temperature, from a probabilistic point of view.

Let us take β large, fix $(\lambda, h) \in \check{U}$, and denote by $\mathscr{G}(\beta,\lambda,h)$ the set of infinite-volume Gibbs measures associated to the Blume–Capel model. As in Chapter 6, the latter are defined as the probability measures compatible with the specification associated to the potential $\beta\Phi = \beta(\Phi^0 + W)$ (remember (7.19)). Let

$$\Upsilon(\beta,\lambda,h) \stackrel{\text{def}}{=} \big\{\# \in \{+,0,-\} : \mathscr{U}_\beta^\# \ni (\lambda,h)\big\}$$

denote the set of stable periodic ground-states at β,λ,h. We will show that, *for each* $\# \in \Upsilon(\beta,\lambda,h)$ *a Gibbs measure* $\mu^\#_{\beta,\lambda,h} \in \mathscr{G}(\beta,\lambda,h)$ *can be prepared using the boundary condition* $\eta^\#$*, under which typical configurations are described by small local perturbations away from* $\eta^\#$. Moreover, these measures are extremal and ergodic. As will be explained in Section 7.6.1, this construction yields the *complete* phase diagram: any other translation-invariant Gibbs measure can be written as a convex combination of the measures $\mu^+_{\beta,\lambda,h}$, $\mu^0_{\beta,\lambda,h}$ and $\mu^-_{\beta,\lambda,h}$.

Remark 7.40. Notice that, using the boundary condition # and proceeding as we did in the proof of Theorem 6.26, we can extract from any sequence $\Lambda_n \uparrow \mathbb{Z}^d$ a subsequence $(\Lambda_{n_k})_{k\ge1}$ such that the limit

$$\lim_{k\to\infty} \mu^{\eta^\#}_{\Lambda_{n_k};\beta,\lambda,h}(f)$$

exists for every local function f, thus defining a Gibbs measure. A priori, this measure depends on the subsequence $(\Lambda_{n_k})_{k\ge1}$. ⋄

Below, we show that, when the temperature is sufficiently low and $\# \in \Upsilon(\beta,\lambda,h)$, the thermodynamic limit used to construct this measure does not depend on the sequence $(\Lambda_n)_{n\ge1}$ and can be controlled in a much more precise way.

In order to use the earlier results that rely on the contour representations of configurations, we will use the Gibbs distributions $\mu^\#_{\Lambda;\beta,\lambda,h}$ for the Blume–Capel model, defined as $\mu^\#_{\Lambda;\Phi}$ in Section 7.3, rather than those of Chapter 6.

Theorem 7.41. *There exists β_0 such that, for all $\beta \geq \beta_0$ and all $(\lambda, h) \in \check{U}$, the following holds.*

1. *Let $\# \in \Upsilon(\beta, \lambda, h)$. For all sequence of* c*-connected sets $\Lambda_n \uparrow \mathbb{Z}^d$ and every local function f, the following limit exists*

$$\mu^{\#}_{\beta,\lambda,h}(f) \stackrel{\text{def}}{=} \lim_{n\to\infty} \mu^{\#}_{\Lambda_n;\beta,\lambda,h}(f), \tag{7.98}$$

and defines a Gibbs measure $\mu^{\#}_{\beta,\lambda,h} \in \mathscr{G}(\beta, \lambda, h)$.

2. *The measures $\mu^{\#}_{\beta,\lambda,h}$, $\# \in \Upsilon(\beta, \lambda, h)$, are translation invariant, extremal and ergodic. Moreover, they are distinct, since*

$$\mu^{\#}_{\beta,\lambda,h}(\sigma_0 = \#) \geq 1 - \delta(\beta), \tag{7.99}$$

where $\delta(\beta) > 0$ tends to zero as $\beta \to \infty$.

In particular, two (resp. three) distinct Gibbs measures can be constructed for each pair (λ, h) living on a coexistence line (resp. at the triple point). (Note that a similar statement could also be derived from Theorem 6.91 and the non-differentiability of the pressure.) Geometric properties of the typical configurations under these measures will be described in Theorem 7.44.

Proof of Theorem 7.41: Fix (β, λ, h) and let $\# \in \Upsilon(\beta, \lambda, h)$. To lighten the notations, we omit β, λ and h everywhere in the indices.

▶ *Proof of* (7.98)*:* Let us fix some local function f. We will first show that, for all $n \geq 1$,

$$\left|\mu^{\#}_{\Lambda}(f) - \mu^{\#}_{\Delta}(f)\right| \leq c\|f\|_\infty n^d e^{-\tau' n}, \tag{7.100}$$

whenever $\Lambda, \Delta \Subset \mathbb{Z}^d$ are c-connected and both contain $\mathsf{B}(2n)$. This implies that $(\mu^{\#}_{\Lambda_n}(f))_{n\geq 1}$ is a Cauchy sequence, which proves the existence of the limit in (7.98). We have already seen that this is sufficient to define the measure $\mu^{\#}$; we leave it as an exercise to verify that $\mu^{\#} \in \mathscr{G}(\beta, \lambda, h)$ (adapt the proof of Theorem 6.26). Observe also that (7.100) shows that the limit does not depend on the chosen sequence $(\Lambda_n)_{n\geq 1}$.

We prove (7.100) using a *coupling* of $\mu^{\#}_{\Lambda}$ and $\mu^{\#}_{\Delta}$. Let

$$\Omega_\Lambda \times \Omega_\Delta \stackrel{\text{def}}{=} \left\{(\omega, \omega') : \omega \in \Omega_\Lambda, \omega' \in \Omega_\Delta\right\}.$$

Let n be large enough to ensure that $\mathsf{B}(n)$ contains the support of f and assume Λ and Δ are both c-connected and large enough to contain $\mathsf{B}(2n)$. On $\Omega_\Lambda \times \Omega_\Delta$, define

$$\mathbb{P}^{\#}_{\Lambda,\Delta} \stackrel{\text{def}}{=} \mu^{\#}_{\Lambda} \otimes \mu^{\#}_{\Delta}.$$

We call $D \subset \mathbb{Z}^d$ **#-surrounding** if (i) D is connected, (ii) $\mathsf{B}(n) \subset D \subset \mathsf{B}(2n)$, (iii) $\omega_i = \omega'_i = \#$ for all $i \in \partial^{\text{ex}} D$.

Let us consider the event $C_{n,\#,\#} \subset \Omega_\Lambda \times \Omega_\Delta$ defined as follows: $(\omega, \omega') \in C_{n,\#,\#}$ if and only if there exists at least one #-surrounding set. Observe that if D_1 and D_2 are #-surrounding, then $D_1 \cup D_2$ is also #-surrounding. When $C_{n,\#,\#}$ occurs, we will therefore denote the *largest* (with respect to inclusion) #-surrounding set by $D_\#$. We denote by $[D_\#]$ the event that $C_{n;\#,\#}$ occurs and $D_\#$ is the largest #-surrounding set. Letting $F(\omega,\omega') \stackrel{\text{def}}{=} f(\omega) - f(\omega')$,

$$\big|\mu^\#_\Lambda(f) - \mu^\#_\Delta(f)\big| = \big|\mathbb{E}^\#_{\Lambda,\Delta}[F]\big| \le \big|\mathbb{E}^\#_{\Lambda,\Delta}[F\mathbf{1}_{C_{n,\#,\#}}]\big| + 2\|f\|_\infty \mathbb{P}^\#_{\Lambda,\Delta}(C_{n,\#,\#}{}^{\text{c}}).$$

For the first term in the right-hand side, we can sum over the possible $D_\#$ (for simplicity, we denote the realization of each such set also by $D_\#$). Note that, by construction, it is sufficient to look at the configuration outside $D_\#$ to determine whether the event $[D_\#]$ occurs; in other words, $[D_\#] \in \mathscr{F}_{D^{\text{c}}_\#}$. Therefore,

$$\mathbb{E}^\#_{\Lambda,\Delta}[F\mathbf{1}_{[D_\#]}] = \mathbb{E}^\#_{\Lambda,\Delta}\big[\mathbb{E}^\#_{\Lambda,\Delta}[F \,|\, \mathscr{F}_{D^{\text{c}}_\#}]\mathbf{1}_{[D_\#]}\big] = 0,$$

since $\mathbb{P}^\#_{\Lambda,\Delta}\big((\omega_{D_\#},\omega'_{D_\#}) \;\big|\; \mathscr{F}_{D^{\text{c}}_\#}\big) = \mu^\#_{D_\#}(\omega_{D_\#})\mu^\#_{D_\#}(\omega'_{D_\#})$ on $[D_\#]$. This implies that

$$\mathbb{E}^\#_{\Lambda,\Delta}[F\mathbf{1}_{C_{n,\#,\#}}] = \sum_{D_\#}\mathbb{E}^\#_{\Lambda,\Delta}[F\mathbf{1}_{[D_\#]}] = 0.$$

We now show that, at low temperature, #-surrounding sets exist with probability close to 1.

Lemma 7.42. *When β is sufficiently large, there exists $\tau' = \tau'(\beta) > 0$ such that*

$$\mathbb{P}^\#_{\Lambda,\Delta}(C_{n,\#,\#}{}^{\text{c}}) \le 2|\mathsf{B}(n)|e^{-\tau' n}. \tag{7.101}$$

Proof. Let $\mathscr{P}_n$ denote the family of all self-avoiding paths $(i_0, i_1, \dots, i_k)$ inside $\mathsf{B}(2n)\setminus\mathsf{B}(n)$, with $i_0 \in \partial^{\text{ex}}\mathsf{B}(n)$, $i_k \in \partial^{\text{in}}\mathsf{B}(2n)$, $i_j \sim i_{j+1}$. We say that i is **(#,#)-correct** if it is #-correct both in ω and in ω'. We claim that

$$C_{n,\#,\#}{}^{\text{c}} \subset \big\{\exists(i_0,\dots,i_k) \in \mathscr{P}_n \text{ such that each } i_j \text{ is not } (\#,\#)\text{-correct}\big\}. \tag{7.102}$$

Indeed, assume that each $\pi = (i_0,\dots,i_k) \in \mathscr{P}_n$ is such that i_j is $(\#,\#)$-correct for at least one index $j \in \{0,1,\dots,k\}$, and let $j(\pi)$ denote the smallest such index. Consider the truncated path $\tilde\pi \stackrel{\text{def}}{=} (i_0, i_1, \dots, i_{j(\pi)})$. Then, clearly, $D \stackrel{\text{def}}{=} \mathsf{B}(n) \cup \bigcup_{\pi\in\mathscr{P}_n}\tilde\pi$ is #-surrounding.

Now, if each vertex of a path $\pi \in \mathscr{P}_n$ is not $(\#,\#)$-correct, then there must exist two collections $\Gamma' = \{\gamma'_1,\dots,\gamma'_l\} \subset \Lambda$ and $\Gamma'' = \{\gamma''_1,\dots,\gamma''_m\} \subset \Delta$ of *external* contours of type # such that

$$\pi \subset \bigcup_{k=1}^{l}\overline{\text{int}\gamma'_k} \cup \bigcup_{k=1}^{m}\overline{\text{int}\gamma''_k},$$

where we introduced the notation $\overline{\text{int}\gamma} \stackrel{\text{def}}{=} \overline{\gamma} \cup \text{int}\gamma$.

- From the collection $\Gamma' \cup \Gamma''$, we can always extract an ordered subcollection $Y = (\gamma_1,\dots,\gamma_k) \subset \Gamma' \cup \Gamma''$ enjoying the following properties: (i) $\overline{\text{int}\gamma_1} \cap \mathsf{B}(n) \ne \varnothing$; (ii) either all contours $\gamma_i \in Y$ whose index i is odd belong to Γ' and those whose index i is even belong to Γ'', or vice versa; (iii) Y is a *chain* in the sense that each $\gamma_i \in Y$ is compatible neither with γ_{i-1} nor with γ_{i+1}, but is compatible with all other γ_js; (iv) if $\overline{Y} \stackrel{\text{def}}{=} \bigcup_{i=1}^k \overline{\gamma}_i$ then $|\overline{Y}| \ge n$ (since π has diameter at least n).

- By construction, Y is made of contours belonging to subcollections $\Gamma_1 \subset \Gamma'$ and $\Gamma_2 \subset \Gamma''$. Using Lemma 7.26 and the fact that all contours of type # are τ-stable when $(\lambda, h) \in \mathscr{U}_\beta^\#$, we get

$$\mu_\Lambda^\#\big(\text{each } \gamma' \in \Gamma_1 \text{ is external}\big) \le \prod_{\gamma' \in \Gamma_1} \mathrm{w}^\#(\gamma') \le \prod_{\gamma' \in \Gamma_1} e^{-\tau|\overline{\gamma}'|}.$$

A similar bound holds for $\mu_\Delta^\#\big(\text{each } \gamma'' \in \Gamma_2 \text{ is external}\big)$.

We can gather these informations into the following bound:

$$\begin{aligned}\mathbb{P}_{\Lambda,\Delta}^\#(C_{n,\#,\#}{}^{\mathrm{c}}) &\le 2 \sum_{k \ge 1} \sum_{\substack{Y=(\gamma_1,\cdots,\gamma_k):\\ \overline{\mathrm{int}\gamma_1} \cap \mathsf{B}(n) \neq \varnothing \\ |\overline{Y}| \ge n}} \prod_{i=1}^k e^{-\tau|\overline{\gamma}_i|} \\ &\le 2e^{-\tau' n} \sum_{k \ge 1} \sum_{\substack{Y=(\gamma_1,\cdots,\gamma_k):\\ \overline{\mathrm{int}\gamma_1} \cap \mathsf{B}(n) \neq \varnothing}} \prod_{i=1}^k e^{-\tau'|\overline{\gamma}_i|}, \end{aligned} \tag{7.103}$$

where $\tau' \stackrel{\text{def}}{=} \tau/2$. We sum over $Y = (\gamma_1, \dots, \gamma_k)$, starting with γ_k:

$$\sum_{\gamma_k : \gamma_k \not\sim \gamma_{k-1}} e^{-\tau'|\overline{\gamma}_k|} \le 3^d |\overline{\gamma}_{k-1}| \sum_{\gamma_k : \overline{\gamma}_k \ni 0} e^{-\tau'|\overline{\gamma}_k|} \le e^{3^d |\overline{\gamma}_{k-1}|} \sum_{\gamma_k : \overline{\gamma}_k \ni 0} e^{-\tau'|\overline{\gamma}_k|}.$$

Then, for $j = k-1, k-2, \dots, 2$,

$$\sum_{\gamma_j : \gamma_j \not\sim \gamma_{j-1}} e^{-(\tau' - 3^d)|\overline{\gamma}_j|} \le e^{3^d|\overline{\gamma}_{j-1}|} \sum_{\gamma_j : \overline{\gamma}_j \ni 0} e^{-(\tau' - 3^d)|\overline{\gamma}_j|}.$$

In the end, we are left with $j = 1$:

$$\sum_{\gamma_1 : \overline{\mathrm{int}\gamma_1} \cap \mathsf{B}(n) \neq \varnothing} e^{-(\tau' - 3^d)|\overline{\gamma}_j|} \le |\mathsf{B}(n)| \sum_{\gamma_1 : \overline{\mathrm{int}\gamma_1} \ni 0} e^{-(\tau' - 3^d)|\overline{\gamma}_j|}.$$

This last sum can be bounded as in Lemma 7.30 and shown to be smaller than some $\eta_1 = \eta_1(\tau', \ell_0) < 1/2$ if τ is large enough. In particular, $\sum_{k \ge 1} \eta_1^k < 1$ and the conclusion follows. □

We have proved (7.100).

► *Proof of translation invariance:* That $\mu^\#$ is translation invariant can be shown exactly as in the proof of Theorem 3.17 (p. 102): for any translation θ_i and any local function f (remember Figure 3.8),

$$|\mu_{\Lambda_n}^\#(f) - \mu_{\Lambda_n}^\#(f \circ \theta_i)| = |\mu_{\Lambda_n}^\#(f) - \mu_{\theta_i \Lambda_n}^\#(f)|,$$

and by (7.100), the right-hand side converges to zero.

► *Proof of extremality:* We will use the characterization of extremality given in item 4 of Theorem 6.58. Let $A \in \mathscr{C}$ be any cylinder, and n be so large that $A \in \mathscr{F}_{\mathsf{B}(n)}$. Let also $B \in \mathscr{F}_{\Lambda^c}$, where Λ is large enough to contain $\mathsf{B}(4n)$. Consider the event $C_{2n,\#} \subset \Omega$ that there exists a largest connected set $D_\#$ such that $\mathsf{B}(2n) \subset D_\# \subset \mathsf{B}(4n)$ and $\omega_i = \#$ for each $i \in \partial^{\mathrm{ex}} D_\#$. Using a decomposition similar to the one used earlier,

$$\mu^\#(A \cap B) = \sum_{D_\#} \mu^\#(A \cap B \cap [D_\#]) + \mu^\#(A \cap B \cap C_{2n,\#}{}^{\mathrm{c}}).$$

Since $C_{2n,\#}$ is local, $\mu^\#(C_{2n,\#}{}^c) = \lim_{N\to\infty} \mu^\#_{\Lambda_N}(C_{2n,\#}{}^c)$. Using Lemma 7.42, for all $N \geq 4n$,

$$\mu^\#_{\Lambda_N}(C_{2n,\#}{}^c) \leq \mathbb{P}^\#_{\Lambda_N,\Lambda_N}(C_{2n,\#,\#}{}^c) \leq 2|\mathsf{B}(2n)|e^{-2\tau' n}. \tag{7.104}$$

Therefore, $\mu^\#(A \cap B \cap C_{2n,\#}{}^c) \leq 2|\mathsf{B}(2n)|e^{-2\tau' n}$. Now, for a fixed $D_\#$, $\mathscr{F}_{\Lambda^c} \subset \mathscr{F}_{D_\#^c}$, and so

$$\mu^\#(A \cap B \cap [D_\#]) = \mu^\#\big(\mu^\#(A \cap B \,|\, \mathscr{F}_{D_\#^c})\mathbf{1}_{[D_\#]}\big) = \mu^\#\big(\mu^\#(A \,|\, \mathscr{F}_{D_\#^c})\mathbf{1}_{B\cap[D_\#]}\big).$$

Since $\mu^\# \in \mathscr{G}(\beta,\lambda,h)$, we have, on $[D_\#]$, $\mu^\#(A \,|\, \mathscr{F}_{D_\#^c}) = \mu^{\eta^\#}_{D_\#}(A)$ almost surely. Adapting (7.106) below gives

$$\mu^{\eta^\#}_{D_\#}(A) = \mu^\#(A) + O(n^d e^{-\tau' n}).$$

Altogether, we get

$$\mu^\#(A \cap B) = \mu^\#(A)\mu^\#(B) + O(n^d e^{-\tau' n}). \tag{7.105}$$

This implies that $\mu^\#$ is extremal.

► *Proof of ergodicity:* Ergodicity follows from extremality and translation invariance, exactly as in the proof of Lemma 6.66.

► *Proof of* (7.99)*:* We reformulate Peierls' argument. We fix some $\Lambda \Subset \mathbb{Z}^d$ and observe that, in any configuration $\omega \in \Omega^\#_\Lambda$ such that $\omega_0 \neq \#$, there exists an external contour $\gamma' \subset \Lambda$ such that $\overline{\text{int}\gamma'} \ni 0$. We can then use again Lemma 7.26 and the stability of the weight of contours of type # when $(\lambda, h) \in \mathscr{U}^\#_\beta$ to obtain, uniformly in Λ, a bound involving the same sum as before:

$$\mu^\#_\Lambda\big(\sigma_0 \neq \#\big) \leq \sum_{\gamma' : \overline{\text{int}\gamma'} \ni 0} e^{-\tau|\overline{\gamma}'|}.$$

This sum can be made arbitrarily small when β (and hence τ) is large enough. This concludes the proof of Theorem 7.41. □

In the above proof, we have actually established more, namely **exponential relaxation** and **exponential mixing** at low temperature.

Corollary 7.43. *Under the same hypotheses as Theorem 7.41, if* $\# \in \Upsilon(\beta,\lambda,h)$*, there exists* $c < \infty$ *such that, for any function* f *having its support inside* $\mathsf{B}(n)$ *and for all* c*-connected* $\Lambda \supset \mathsf{B}(2n)$*,*

$$\big|\mu^\#_{\Lambda;\beta,\lambda,h}(f) - \mu^\#_{\beta,\lambda,h}(f)\big| \leq c\|f\|_\infty n^d e^{-\tau' n}. \tag{7.106}$$

Moreover, for any $\mathscr{F}_{\mathsf{B}(4n)^c}$*-measurable function* g*,*

$$\big|\mu^\#_{\beta,\lambda,h}(fg) - \mu^\#_{\beta,\lambda,h}(f)\mu^\#(g)_{\beta,\lambda,h}\big| = c\|f\|_\infty\|g\|_\infty n^d e^{-\tau' n}.$$

Proof. The first claim follows by taking $\Delta \uparrow \mathbb{Z}^d$ in (7.100). The second follows from the same argument that led to (7.105). □

A characterization of "the sea of # with small islands". The bound (7.99) suggests that a typical configuration under $\mu^{\#}_{\beta,\lambda,h}$ displays only small local deviations from the ground state $\eta^{\#}$. Here, we will provide a more global characterization, by giving a description of configurations on the whole lattice, that holds almost surely.

For instance, (7.104) implies that, for all $\# \in \Upsilon(\beta,\lambda,h)$,

$$\sum_n \mu^{\#}_{\beta,\lambda,h}({C_{n,\#}}^{\mathrm{c}}) < \infty .$$

Therefore, the Borel–Cantelli Lemma implies that, $\mu^{\#}_{\beta,\lambda,h}$-almost surely, all but a finite number of the events $C_{n,\#}$ occur simultaneously. This means that the origin is always surrounded by an infinite number of #-surrounding sets, of arbitrarily large sizes. But it does not yet rule out the presence of $\#'$-surrounding sets, for other labels $\#'$.

To remedy this problem, let $N > n$ and consider $E_{N,n,\#} \stackrel{\text{def}}{=} F_{N,n,\#} \cap C_{n,\#}$, where $C_{n,\#}$ was defined earlier and $F_{N,n,\#}$ is the event that there exists a self-avoiding path $\pi = (i_0, i_1, \dots, i_k) \subset \mathrm{B}(N) \setminus \mathrm{B}(n)$, with $i_0 \in \partial^{\text{ex}}\mathrm{B}(n)$, $i_k \in \partial^{\text{in}}\mathrm{B}(N)$, $i_j \sim i_{j+1}$, such that $\omega_{i_j} = \#$ for all j. On the event

$$E_{n,\#} \stackrel{\text{def}}{=} \bigcap_{N>n} E_{N,n,\#} ,$$

there exists a #-surrounding $\mathrm{B}(n) \subset D_\# \subset \mathrm{B}(2n)$, and there exists an infinite self-avoiding path π (connecting $D_\#$ to $+\infty$) of vertices i with $\omega_i = \#$ (see Figure 7.10).

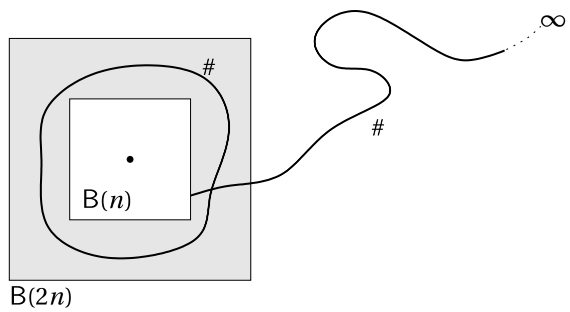


Figure 7.10: Almost surely under $\mu^{\#}_{\beta,\lambda,h}$, the origin (as well as every other vertex of the lattice) is surrounded by a circuit (in $d = 2$; otherwise, a closed surface) of #-spins, and this circuit is itself connected to $+\infty$ by a path of #-spins.

Theorem 7.44. *Let $\# \in \Upsilon(\beta,\lambda,h)$. Then,*

$$\mu^{\#}_{\beta,\lambda,h}\big(\exists M < \infty \text{ such that } E_{M,\#} \text{ occurs}\big) = 1 .$$

Proof. Let us study $F_{N,n,\#}$ under $\mu^{\#}_{\Lambda;\beta,\lambda,h}$. Observe that

$${F_{N,n,\#}}^{\mathrm{c}} \subset \big\{\text{there exists an external contour } \gamma' \subset \Lambda \text{ such that } \overline{\operatorname{int}\gamma'} \supset \mathrm{B}(n)\big\} .$$

Now, if $\overline{\text{int}\gamma'} \supset \mathsf{B}(n)$, then $|\overline{\gamma}'| \geq |\partial^{\text{ex}}\mathsf{B}(n)| \geq n^{d-1}$. Therefore, we can proceed as earlier to obtain, for all c-connected $\Lambda \supset \mathsf{B}(N)$,

$$\begin{aligned}\mu^{\#}_{\Lambda;\beta,\lambda,h}(F_{N,n,\#}{}^{\mathrm{c}}) &\leq \sum_{\substack{\gamma':\overline{\text{int}\gamma'}\cap\mathsf{B}(n)\neq\varnothing\\ |\overline{\gamma}'|\geq n^{d-1}}} e^{-\tau|\overline{\gamma}'|}\\ &\leq |\mathsf{B}(n)| \sum_{\substack{\gamma':\overline{\text{int}\gamma'}\ni 0\neq\varnothing\\ |\overline{\gamma}'|\geq n^{d-1}}} e^{-\tau|\overline{\gamma}'|} \leq |\mathsf{B}(n)|e^{-\tau'' n^{d-1}},\end{aligned}$$

uniformly in N and Λ, for some $\tau'' > 0$ depending on β. Since $E_{N,n,\#}$ is decreasing in N,

$$\mu^{\#}_{\beta,\lambda,h}(E_{n,\#}) = \lim_{N\to\infty} \mu^{\#}_{\beta,\lambda,h}(E_{N,n,\#}) \geq 1 - \epsilon_n,$$

where $\epsilon_n \stackrel{\text{def}}{=} |\mathsf{B}(n)|(2e^{-\tau' n} + e^{-\tau'' n^{d-1}})$. Since ϵ_n is summable, we can again apply the Borel-Cantelli lemma to conclude that all but a finite number of events $E_{n,\#}$ occur $\mu^{\#}_{\beta,\lambda,h}$-almost surely. □

Combining the previous result with translation invariance, we summarize the almost-sure properties of typical configurations in a theorem:

Theorem 7.45. *Let β be large enough and $(\lambda, h) \in \mathscr{U}^{\#}_{\beta}$. For all $\# \in \Upsilon(\beta, \lambda, h)$, under $\mu^{\#}_{\beta,\lambda,h}$, a typical configuration consists in a* ***sea of # (the ground state $\eta^{\#}$) with local bounded deformations****, in the following sense: every vertex $i \in \mathbb{Z}^d$ is either connected to $+\infty$ by a self-avoiding path along which all spins are #, or there exists a finite external contour γ such that $\overline{\text{int}\gamma} \ni i$.*

The study of the largest contours in a box can be done as for the Ising model:

Exercise 7.15. *Let β be large, as above. Fix $\# \in \Upsilon(\beta, \lambda, h)$, and consider the Blume–Capel in $\mathsf{B}(n)$. Adapting the method of Exercise 3.18, show that under $\mu^{\#}_{\mathsf{B}(n);\beta,\lambda,h}$, the largest contours in $\mathsf{B}(n)$ have a support of size of order* $\log n$.

7.5 Bibliographical references

Although it is sometimes unfairly referred to as a “generalization of Peierls’ argument”, the Pirogov–Sinai theory (PST) actually uses several important concepts of equilibrium statistical mechanics and introduces important new ideas. The original method introduced by Pirogov and Sinai (English translations of their original papers can be found in [313]) was based on the use of contours and of a Hamiltonian satisfying Peierls’ condition, but the phase diagram was constructed by using an abstract approach involving a fixed-point argument. Later, Zahradník [354] contributed substantially to the theory by introducing fundamental new ideas. In particular, he introduced the notion of truncated pressure, which eventually superseded the original fixed-point argument and became the core of the current understanding of the theory. For that reason, it would be more correct to call it the *Pirogov–Sinai–Zahradník theory.*

Pedagogical texts on PST include the paper [34] by Borgs and Imbrie, and the lecture notes by Fernández [104]. The review paper of Slawny [316], although based on the fixed-point method of Pirogov and Sinai, is clear and applies the theory to various models.

Our Section 7.2, devoted to ground states, is inspired by the presentation in Chapter 2 of Sinai's book [312]. Other notions of ground states exist in the literature (see [85]). The method to determine the set of periodic ground states based on m-potentials, as presented in Section 7.2.2, is due to [164]. An interesting account of the main ideas of PST as well as a description of more general notions of ground state can be found in [343].

The model considered in Example 7.8 is due to Pechersky in [265]. Prior to that counter-example, it had been conjectured [312] that a finite-range potential with a finite number of periodic ground states would always satisfy Peierls' condition.

Our analysis of the phase diagram is based on the ideas introduced by Zahradník and followers. In particular, the C^1-truncation used when defining the truncated weights is a simpler version of the C^k-truncations used by Borgs and Kotecký in [35].

Although the details were only implemented for the Blume–Capel model, the methods used are general and can be applied in many other situations. As an exercise, the interested reader can use them to provide a full description of the low-temperature phase diagram of the modified Ising model described early in Section 7.1.1.

7.6 Complements and further reading

7.6.1 Completeness of the phase diagram

One of the main results in this chapter was the construction of low-temperature translation-invariant extremal Gibbs measures for the Blume–Capel model using stable periodic (actually constant) ground states as boundary conditions: $\mu^{\#}_{\beta,\lambda,h}$, $\# \in \Upsilon(\beta,\lambda,h)$.

At this stage, it is very natural to wonder whether there are other Gibbs measures, in addition to those constructed here. By the general theory of Chapter 6, the set of infinite-volume Gibbs measures is a simplex, so we can restrict our discussion to extremal measures. The following remarkable result shows that the Gibbs measure we constructed exhaust the set of translation-invariant Gibbs measures: at sufficiently low temperature, any translation-invariant measure in $\mathscr{G}_\theta(\beta,\lambda,h)$ can be represented as a convex combination of $\mu^{\#}_{\beta,\lambda,h}$, $\# \in \Upsilon(\beta,\lambda,h)$.

Theorem 7.46. *There exists β_0 such that, for all $\beta \geq \beta_0$, the following holds. For all $(\lambda,h) \in \check{U}$ and all $\mu \in \mathscr{G}_\theta(\beta,\lambda,h)$, there exist coefficients $(\alpha_\#)_{\#\in\Upsilon(\beta,\lambda,h)} \subset [0,1]$ such that*

$$\mu = \sum_{\#\in\Upsilon(\beta,\lambda,h)} \alpha_\# \mu^{\#}_{\beta,\lambda,h}\,.$$

In particular, in the regions where only one of the boundary conditions $+,-,0$ is stable (the interior of the regions $\mathscr{U}^{\#}_{\beta}$ on Figure 7.2), there is a *unique* translation-invariant Gibbs measure.

The same statement holds for the general class of models to which the theory applies; see the original paper of Zahradník [354], where a proof can be found. This kind of statement is usually referred to as the **completeness of the phase diagram.**

We therefore see that, while the Pirogov–Sinai Theory is limited to perturbative regimes (here, very low temperature), it provides in such regimes a complete description of the set of all translation-invariant Gibbs measures. Of course, there are, in general, other non-translation-invariant Gibbs measures, such as Dobrushin states in the Ising model in dimensions $d \geq 3$ (see the discussion in Section 3.10.7). Extensions of PST dealing with such states have been developed; the reader can consult, for instance, [162].

7.6.2 Generalizations

PST has been extended in various directions, for instance to systems with continuous spins [87, 48], quasiperiodic interactions [199], or long-range interactions [262, 263].

One important application of PST was in the seminal work of Lebowitz, Mazel and Presutti [214], in which a first-order phase transition is proved for a model of particles in the continuum with Kac interactions (of finite range), similar to those we considered in Section 4.10. At the core of their technique lies a non-trivial definition of *contour* associated to configurations of point particles in the continuum, and the use of the main ideas of PST.

7.6.3 Large-β asymptotics of the phase diagram

The analysis of this chapter showed that the low-temperature phase diagram of the Blume–Capel model is a small perturbation of the corresponding one at zero temperature. A more delicate analysis is required if one wants to derive more *quantitative* information on this diagram as a function of β.

For instance, $(\lambda, h) = (0,0)$ is the triple point of the phase diagram at zero temperature, at which η^+, η^0 and η^- are ground states. The following question is natural: in which direction does the triple point move when the inverse temperature is finite? In other words: which are the stable phases at $(0,0)$ when $\beta < \infty$?

In principle, this question can be answered by determining which truncated pressure $\widehat{\psi}^\#(0,0) = \hat{g}^\#(0,0)$ (remember that $\mathrm{e}^\#(0,0) = 0$), $\# \in \{+,0,-\}$, is maximal. But, each $\hat{g}^\#(0,0)$ is a series made of products of τ-stable weights, where τ can be made large when β is large. Computing the first terms of these expansions should allow to determine which one dominates. Unfortunately, the structure of the truncated weights makes extracting such information difficult. We briefly describe an alternative approach, informally, providing references for the interested reader.

Let us describe the contributions from the smallest perturbations of the ground state $\eta^\#$, which provide the main contribution to $\hat{g}^\#$. We do this at a heuristic level.

Consider first the case $\# = +$. Among the configurations that coincide with η^+ everywhere except on a finite set, the configurations with lowest energy are those which have a single 0 spin. The energy associated to such an *excitation* is $2d$. It therefore seems plausible that the leading term in the expansion of $\hat{g}^+(0,0)$ is due to the cluster made of exactly one such excitation, that is

$$\beta\widehat{\psi}^+(0,0) = e^{-2d\beta} + \dots,$$

where the dots stand for higher order terms in $e^{-\beta}$.

Now, among the configurations that coincide with η^0 everywhere except on a finite set, the configurations with lowest energy are those which have either a single

+ spin or a single − spin. Both these excitations have an energy equal to $2d$, as before, which leads to

$$\beta\widehat{\psi}^0(0,0) = 2e^{-2d\beta} + \dots .$$

Provided the higher order terms yield significantly smaller contributions, this gives, at large β,

$$\widehat{\psi}^0(0,0) - \widehat{\psi}^+(0,0) = \frac{1}{\beta}e^{-2d\beta} + \dots \geq \frac{1}{2\beta}e^{-2d\beta} > 0\,,$$

implying that only the 0 phase is stable at $(0,0)$. In other words: the triple point shifts to the *right* at positive temperatures, as depicted in Figure 7.9.

In fact, this argument can be repeated when (λ, h) lies in a neighborhood of $(0,0)$, allowing to construct the coexistence line $\mathscr{L}_\beta^{0\pm}$. Namely, fix β large and take λ, h such that $\beta\lambda \ll 1$ and $\beta h \ll 1$. Arguing as above, considering the excitations of smallest energy, we get (remember that $\mathrm{e}^+(\lambda, h) = \lambda + h$)

$$\beta\widehat{\psi}^0(\lambda,h) = e^{-2d\beta+\beta\lambda+\beta h} + e^{-2d\beta+\beta\lambda-\beta h} + \dots ,$$
$$\beta\widehat{\psi}^+(\lambda,h) = \beta\lambda + \beta h + e^{-2d\beta-\beta\lambda-\beta h} + \dots ,$$

Therefore, for very large β, the smooth map describing $\mathscr{L}_\beta^{0+}$ (see Figure 7.8) can be obtained, in first approximation, by equating these two expressions, yielding

$$\lambda \mapsto h(\lambda) = \beta^{-1}e^{-2d\beta} - \lambda\bigl(1 - 4e^{-2d\beta}\bigr) + O(\lambda^2)\,.$$

In particular, the position of the triple point $(\lambda_*(\beta), 0)$, is obtaining by solving $h(\lambda) = 0$, which yields

$$\lambda_*(\beta) = \frac{e^{-2d\beta}}{\beta}(1 + O(e^{-2d\beta}))\,.$$

These computations are purely formal, but their conclusions can be made rigorous. The idea is to replace the notion of ground state by the notion of **restricted ensemble**. In the context described above, the restricted ensemble $\mathscr{R}^0$ is defined as the set of all configurations ω such that $\omega_i \neq 0 \implies \omega_j = 0$ for all $j \sim i$. That is, they correspond to the ground state η^0 on top of which only the smallest possible excitations are allowed. Starting from a general configuration, one then erases all such smallest energy excitations and construct contours for the resulting configuration. Of course, this is more delicate than before, since, in contrast to the ground state η^0, the restricted phase $\mathscr{R}^0$ has a nontrivial pressure that, in particular, depends on the volume. Nevertheless, the analysis can be done along similar lines. We refer to the lecture notes [50] by Bricmont and Slawny for a pedagogical introduction to this problem (and a proof that the triple point of the Blume–Capel model is indeed shifted to the right) and to their paper [51] for a more detailed account.

7.6.4 Other regimes.

Generically, the methods of PST can be used to study models whose partition function can be written as a system of contours, with some equivalent of Peierls' condition. The perturbation parameter need not be the temperature, and the parameter driving the transition need not be related to some external field as we saw in the Blume–Capel model. Consider, for example, the Potts model with spins $\omega_i \in \{0,1,2,\dots,q-1\}$, at inverse temperature β, and denote its pressure by $\psi_q(\beta)$. It turns out that, when q is large, q^{-1} can be used as a perturbation parameter to study $\beta \mapsto \psi_q(\beta)$. Using the methods of PST, it was shown in [48, 233, 203] that a first-order phase transition in β occurs when q is large enough: there exists $\beta_c = \beta_c(q)$ such that the pressure is differentiable when $\beta < \beta_c$ and $\beta > \beta_c$, and that it is non-differentiable at β_c. (This result was first proved, using reflection positivity, in [197]. In two dimensions, the simplest proof, relying on a variation of Peierls' argument, can be found in [93].)

In addition to perturbations of a finite collection of ground states, PST can also be applied successfully to analyze perturbations of other well-understood regimes, usually involving constraints of a certain type. This appears for instance in the study of Kac potentials in the neighborhood of mean-field [214], or the use of restricted phases as in [48, 50, 113].

7.6.5 Finite-size scaling.

In [35], Borgs and Kotecký used the ideas of PST to initiate a theory of *finite-size scaling*, that is, a thorough analysis of the *rate* at which certain thermodynamic quantities (the magnetization, for example) converge to their asymptotic values in the thermodynamic limit. In addition to its obvious theoretical interest, such an analysis also plays an essential role when extrapolating to infinite systems the information obtained from the observation of the relatively small systems that can be analyzed using numerical simulations.

7.6.6 Complex parameters, Lee–Yang zeroes and singularities.

With minor changes, most of the material presented in this chapter can be extended to include complex fields; see [34], for example.

As an interesting application, it has been shown in [20] and [21] that the Lee–Yang theory, exposed for the Ising model in Section 3.7.3, can be extended to other models, allowing to determine the locus of the zeros of their partition function. Of course, since they rely on the main results of PST, these results hold only in a perturbative regime.

Furthermore, the techniques of PST can be used to obtain finer analytic properties of the pressure. Consider for instance the Ising model in a complex magnetic field $h \in \mathbb{C}$. For simplicity, let us consider the contours defined in Section 5.7.4. The magnetic field leads one to introduce contours of two types: + and −. Then, using the trick (7.31), one is led to two types of weights: $\mathrm{w}^+(\gamma)$ and $\mathrm{w}^-(\gamma)$. When $h = 0$, these coincide with those defined in (5.41), but otherwise they contain ratios of partition functions. The weight of a contour of type +, for example, takes the form

$$\mathrm{w}^+(\gamma) = e^{-2\beta|\gamma|}\frac{\mathbf{Z}^-(\mathrm{int}_-\gamma)}{\mathbf{Z}^+(\mathrm{int}_-\gamma)} = e^{-2\beta|\gamma|}\frac{e^{-\beta h|\mathrm{int}_-\gamma|}\Xi^-(\mathrm{int}_-\gamma)}{e^{+\beta h|\mathrm{int}_-\gamma|}\Xi^+(\mathrm{int}_-\gamma)}.$$

The analysis can then be done following the main induction used earlier for the Blume–Capel model. When the field is real, the symmetry between + and − implies that all weights are stable at $h = 0$. For general complex values of h, the symmetry implies that all weights are stable on the imaginary axis $\{\mathfrak{Re}\, h = 0\}$. The weights are then shown to be well defined and analytic in regions of the complex plane analogous to the stability regions defined earlier; if γ is of type +, its weight is analytic in

$$\mathscr{U}_\gamma^+ \stackrel{\text{def}}{=} \Big\{\mathfrak{Re}\, h > -\frac{\theta}{|\mathrm{int}_-\gamma|^{1/d}}\Big\},$$

for a suitable constant θ. Implementing this analysis allows one to obtain a more quantitative version of the Lee–Yang Theorem (restricted to low temperature).

Then, in a second step, Isakov's analysis [174] provides estimate of high-order derivatives of the pressure (see Sections 3.10.9 and 4.12.3), showing that the function $h \mapsto \psi_\beta(h)$ cannot be analytically continued through $h = 0$, along either of the real paths $h \downarrow 0$, $h \uparrow 0$. This analysis was generalized by Friedli and Pfister [114] to all two-phase models to which PST applies, which implies in particular that the pressure of the Blume–Capel model has no analytic continuation accross the lines of coexistence of its phase diagram, at least for low temperatures, away from the triple point.