The effect of numerical integration in the finite element method for nonmonotone nonlinear elliptic problems with application to numerical homogenization methods

Assyr Abdulle, Gilles Vilmart
Section de Mathématiques, École Polytechnique Fédérale de Lausanne, Station 8, 1015 Lausanne, Switzerland.

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Abstract

A finite element method with numerical quadrature is considered for the solution of a class of second-order quasilinear elliptic problems of nonmonotone type. Optimal a-priori error estimates for the H^1 and the L^2 norms are derived. The uniqueness of the finite element solution is established for a sufficiently fine mesh. Our results permit the analysis of numerical homogenization methods. *To cite this article: A. Abdulle, G. Vilmart, C. R. Acad. Sci. Paris, Ser. I ** (20**).*

Résumé

L'effet de l'intégration numérique sur la méthode des éléments finis pour des problèmes non-monotones elliptiques, avec application aux méthodes numériques d'homogénéisation. On considère des méthodes d'éléments finis avec intégration numérique par quadrature pour des problèmes elliptiques quasi-linéaires de type non-monotone. Les vitesses de convergence optimales pour les normes H^1 et L^2 sont démontrées ainsi que l'unicité de la solution numérique pour un maillage suffisamment fin. Ces résultats permettent l'analyse multi-échelles de méthodes d'homogénéisation numérique.

Version française abrégée

Pour des problèmes elliptiques linéaires ou monotones, l'effet de l'intégration numérique sur la méthode des éléments finis est analysé dans [7,15] et [12]. Cependant, il n'existe à notre connaissance aucune analyse de vitesses de convergence pour des problèmes nonlinéaires de type nonmonotones. Dans [11], la convergence H^1 de la solution numérique est établie, mais sans vitesse de convergence et seulement pour des éléments finis linéaires par morceaux. L'objet de cet article est d'analyser l'influence des erreurs de quadrature pour la méthode des éléments finis appliquée à la classe d'équations elliptiques quasi-linéaires non-monotones (1). Sous des hypothèses usuelles sur le maillage pour des problèmes non-linéaires, sur la régularité des coefficients et des données, et sur les formules de quadrature (**Q1**),(**Q2**), également usuelles tant pour des problèmes avec intégration numérique (voir [7] ou [8, Sect. 29]) que pour des problèmes non-linéaires [11,16,10], nous prouvons des estimations optimales d'erreur pour les normes H^1 et L^2 de la méthode d'éléments finis (4), pour des élements simpliciaux ou quadrilatéraux d'ordre arbitraire. Nous prouvons également l'unicité de la solution numérique.

Une application importante de notre étude est une analyse (avec discrétisation totale des échelles à la fois macroscopiques et microscopiques) d'une méthode d'homogénéisation numérique du type [1,2,3,10]

Email addresses: assyr.abdulle@epfl.ch (Assyr Abdulle), gilles.vilmart@epfl.ch (Gilles Vilmart).

pour une classe de problèmes non-linéaires d'homogénisation. La méthode d'homogénéisation numérique considérée peut être interprétée comme une méthode des éléments finis mise en œuvre sur un schéma macroscopique, avec intégration numérique en la variable macroscopique, couplée à des schémas microscopiques mis en œuvre sur des micro-cellules contenues dans le maillage macroscopique. Pour la classe de problèmes nonmonotones (13), il n'existait jusqu'alors qu'une analyse semi-discrete et pour les dimensions $d \leq 2$. Notre analyse, avec discrétisation totale, permet de traiter la dimension $d \leq 3$. De plus, nous proposons une analyse de convergence (optimale) dans la norme L^2. Nous améliorons aussi l'estimation de l'erreur dite de résonnance et démontrons la convergence de la méthode de Newton utilisée pour calculer en pratique une solution du système non-linéaire. Plus de détails sur les résultats et l'analyse présentée ici sont donnés dans [4] (une seule échelle) et [5] (problèmes multi-échelles).

1. Introduction

We study finite element (FE) discretizations of second-order quasilinear elliptic problems of the form

$$-\nabla \cdot (a(x, u(x))\nabla u(x)) = f(x) \text{ in } \Omega, \qquad u(x) = 0 \text{ on } \partial\Omega, \tag{1}$$

where Ω is a bounded polyhedron in $\mathbb{R}^d$ with $d \leq 3$. We make the following assumptions on the tensor $a(x, s) = (a_{mn}(x, s))_{1\leq m,n\leq d}$:

- the coefficients $a_{mn}(x, s)$ are continuous functions on $\overline{\Omega} \times \mathbb{R}$ which are uniformly Lipschitz continuous with respect to s, i.e.,

$$\exists \Lambda_1 > 0, \quad |a_{mn}(x, s_1) - a_{mn}(x, s_2)| \leq \Lambda_1 |s_1 - s_2|, \ \forall x \in \overline{\Omega}, \forall s_1, s_2 \in \mathbb{R}, \forall\ 1 \leq m, n \leq d. \tag{2}$$

- $a(x, s)$ is uniformly coercive and bounded, i.e.,

$$\exists \lambda, \Lambda_0 > 0, \quad \lambda\|\xi\|^2 \leq a(x, s)\xi \cdot \xi, \qquad \|a(x, s)\xi\| \leq \Lambda_0\|\xi\|, \qquad \forall \xi \in \mathbb{R}^d, \forall x \in \overline{\Omega}, \forall s \in \mathbb{R}. \tag{3}$$

Since (2)-(3) hold, it is known [13] that (1) has a unique solution $u \in H^1_0(\Omega)$ for all $f \in L^2(\Omega)$.

For linear or monotone elliptic problems, the effect of numerical quadrature in FEM has been analysed in [7,15] and [12]. To the best of our knowledge, there exist no analysis of the convergence rates for FEM with numerical quadrature applied to nonlinear problems of non-monotone type, as considered in this paper. In [11], the convergence in H^1 of the method is shown for piecewise finite elements, but without convergence rates. In the absence of numerical quadrature, optimal apriori error estimates in the H^1 and L^2 norms for FE methods (FEMs) were first given in [9].

Equations as (1) enter in the modeling of many important problems, we mention the infiltration of water in porous medium, the study of electrical potential or thermal diffusion in materials. As exact integration in FEMs is rarely possible, it is important to quantify the effect of numerical quadrature. Optimal convergence rates in the H^1 and L^2 norms are proved in this case. The practical implementation of the non-linear FEM requires a Newton method. We also establish the convergence of this latter method (crucial in applications) and the uniqueness of the FE solution for a sufficiently fine FE mesh. If $a(x, s)$ becomes independent of s, we recover the results of [7] on FEMs with numerical quadrature for linear problems (polyhedral domain case).

Application to numerical homogenization methods is then considered. In contrast to previous results [10] obtained for nonmonotone homogenisation problems in dimension $d \leq 2$ (based on $2d$-Green function logarithmic estimates) for the H^1 norm and for a semi-discrete formulation, we obtain optimal convergence results for dimensions $d \leq 3$ and for a fully discrete method, which takes into account the microscale FE discretization (see [1,2,3] in the context of linear problems). In addition, our results are also valid for arbitrary high-order elements of simplicial or quadrilateral type, optimal error estimates are obtained for the L^2 norm, and improved estimates are obtained for the resonance error. More details on the results and the analysis presented here are given in [4] (one-scale problems) and [5] (multi-scale problems) .

2. Finite element method with numerical quadrature

We consider a conformal shape regular family of partitions $\mathcal{T}_h$ of Ω in simplicial or quadrilateral elements K of diameter h_K and denote $h := \max_{K\in\mathcal{T}_h} h_K$. We consider the family of FE spaces $S_0^\ell(\Omega,\mathcal{T}_h) := \{v^h \in H_0^1(\Omega);\ v^h|_K \in \mathcal{R}^\ell(K),\ \forall K \in \mathcal{T}_h\}$, where $\mathcal{R}^\ell(K)$ is the space $\mathcal{P}^\ell(K)$ of polynomials on K of total degree at most ℓ if K is a simplicial FE, or the space $\mathcal{Q}^\ell(K)$ of polynomials on K of degree at most ℓ in each variable if K is a quadrilateral FE. We define a quadrature formula $\{\hat{x}_j, \hat{\omega}_j\}_{j=1}^J$ on a reference element $\hat{K}$, where $\hat{x}_j$ are integration points and $\hat{\omega}_j$ are quadrature weights. The quadrature formula $\{x_{K_j}, \omega_{K_j}\}_{j=1}^J$ is then defined as usual on any element K of the triangulation using a C^1-diffeomorphism. We make the following assumptions, which are similar to the case of linear elliptic problems (see [7] or [8, Sect. 29]):

(Q1) $\hat{\omega}_j > 0,\ j = 1,\dots,J,\quad \sum_{j=1}^J \hat{\omega}_j |\nabla\hat{p}(\hat{x}_j)|^2 \geq \hat{\lambda}\|\nabla\hat{p}\|^2_{L^2(\hat{K})},\ \forall\hat{p}(\hat{x}) \in \mathcal{R}^\ell(\hat{K}),\quad$ with $\hat{\lambda} > 0$;

(Q2) $\int_{\hat{K}} \hat{p}(x)dx = \sum_{j=1}^J \hat{\omega}_j \hat{p}(\hat{x}_j),\ \forall\hat{p}(\hat{x}) \in \mathcal{R}^\sigma(\hat{K})$, where $\sigma = \max(2\ell-2,\ell)$ if $\hat{K}$ is a simplicial FE, or $\sigma = \max(2\ell-1,\ell+1)$ if $\hat{K}$ is a rectangular FE.

Consider for v, w scalar or vector functions that are piecewise continuous with respect to the partition $\mathcal{T}_h$ of Ω, the semi-definite inner product $(u,v)_h := \sum_{K\in\mathcal{T}_h}\sum_{j=1}^J \omega_{K_j} u(x_{K_j})v(x_{K_j})$. The FE solution of (1) with numerical integration reads: find $u^h \in S_0^\ell(\Omega,\mathcal{T}_h)$ such that

$$(a(\cdot,u^h)\nabla u^h, \nabla w^h)_h = F_h(w^h) \qquad \forall w^h \in S_0^\ell(\Omega,\mathcal{T}_h), \tag{4}$$

where the linear form $F_h(w^h)$ is an approximation of $\int_\Omega f(x)w^h(x)dx$ obtained for example by using a quadrature formula. If $f \in W^{\ell,r}(\Omega)$ with $1 \leq r \leq \infty$ and $\ell > d/r$, then f is continuous on $\overline{\Omega}$ and one can take $F_h(w^h) := (f, w^h)_h$. The existence of the FE solution $u^h \in S_0^\ell(\Omega,\mathcal{T}_h)$ in (4) can be shown for all $h > 0$ using the Brouwer fixed point theorem. Details can be found for example in [9].

3. Convergence rates for FEM with numerical quadrature for nonlinear problems

Theorem 3.1 *[4] Consider u the solution of problem* (1). *Let $\ell \geq 1$. Let $d/\ell < r \leq \infty$. Let $\mu = 0$ or 1. Assume* (**Q1**)*,* (**Q2**)*, that the family of triangulations is quasi-uniform, and* [1]

$$u \in H^{\ell+1}(\Omega) \cap W^{1,\infty}(\Omega), \qquad a \in (W^{\ell+\mu,\infty}(\Omega\times\mathbb{R}))^{d\times d}, \qquad f \in W^{\ell+\mu,r}(\Omega).$$

In addition to (2), (3)*, assume that the operator $L^*\varphi = -\nabla\cdot(a(\cdot,u)^T\nabla\varphi) + \partial_u a(\cdot,u)\nabla u\cdot\nabla\varphi$ satisfies* [2]

$$\|\varphi\|_{H^2(\Omega)} \leq C(\|L^*\varphi\|_{L^2(\Omega)} + \|\varphi\|_{H^1(\Omega)}), \qquad \textit{for all } \varphi \in H^2(\Omega)\cap H_0^1(\Omega). \tag{5}$$

Assume further that $\partial_u a_{mn} \in W^{1,\infty}(\Omega\times\mathbb{R})$, and that the coefficients $a_{mn}(x,s)$ are twice differentiable with respect to s, with the first and second order derivatives continuous and bounded on $\overline{\Omega}\times\mathbb{R}$, for all $m,n = 1\dots d$.

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1. Except for the $W^{1,\infty}$ assumption on u and the smoothness of $s \mapsto a(x,s)$ assumed to treat the non-linearity (as in [9]), the smoothness assumptions of Thm. 3.1 are identical to those classically assumed for linear problems [7],[8, Sect. 29].

2. The assumption (5) on the adjoint L^* of the linearized operator L associated to (1) is also required for L^2 estimates in the case of linear problems [7]. Using classical H^2 regularity results, it is automatically satisfied –owing to the assumptions on the coefficients of L^*– if the domain Ω is a convex polyhedron

Then there exists $h_0 > 0$ such that for all $h \le h_0$, the solution u^h of (4) is unique and the following H^1 and L^2 error estimates hold,

$$\text{if } \mu = 0,1, \qquad \|u - u^h\|_{H^1(\Omega)} \le Ch^\ell \qquad \text{for all } h \le h_0, \tag{6}$$
$$\text{if } \mu = 1, \qquad \|u - u^h\|_{L^2(\Omega)} \le Ch^{\ell+1} \qquad \text{for all } h \le h_0, \tag{7}$$

where the constant C is independent of h.

Inspired by [9], the proof of Theorem 3.1 is conducted in three main steps.

Step 1. Using the compact injection $H^1(\Omega) \subset L^2(\Omega)$, the boundedness of a numerical solution in $H^1_0(\Omega)$ and the uniqueness in $H^1_0(\Omega)$ of the exact solution of (1), we show,

$$\|u - u^h\|_{L^2(\Omega)} \to 0 \text{ for } h \to 0. \tag{8}$$

Step 2. We derive the following H^1 a-priori error bound

$$\|u - u^h\|_{H^1(\Omega)} \le C(h^\ell + \|u - u^h\|_{L^2(\Omega)}), \text{ for all } h > 0. \tag{9}$$

The additional term $\|u-u^h\|_{L^2(\Omega)}$ in the right-hand side is due to the non-monotonicity of the differential operator of (1). The proof of (9) relies on an estimate for $(a(u^h)\nabla u^h, \nabla w^h) - (a(u^h)\nabla u^h, \nabla w^h)_h$ (obtained by using the Bramble-Hilbert lemma), uniform bounds for the semi-definite inner product $(v,w)_{\mathcal{T}_h} := \sum_{K\in\mathcal{T}_h}\sum_{j=1}^J \omega_{K_j} v(x_{K_j}) \cdot w(x_{K_j})$ (defined for piecewise continuous functions v, w) and the use of the Gagliardo-Niremberg inequality $\|v\|^2_{L^3(\Omega)} \le C\|v\|_{L^2(\Omega)}\|v\|_{H^1(\Omega)}$ for all $v \in H^1(\Omega)$, for $d \le 3$.

Step 3. Using an Aubin-Nitsche duality argument and (5), we show that there exists $h_1 > 0$ such that

$$\|u - u^h\|_{L^2(\Omega)} \le C(h^{\ell+\mu} + \|u - u^h\|^2_{H^1(\Omega)}), \text{ for all } h \le h_1. \tag{10}$$

We consider the FEM solution with numerical quadrature associated to the indefinite linear elliptic problem L^*. We first show that L^* is an isomorphism and then derive error estimates generalizing a compactness result of Schatz [14] to FEM with numerical quadrature.

Proof of the H^1 and L^2 estimates. Substituting (9) into (10) (with $\mu = 0$), we obtain

$$\|u - u^h\|_{H^1(\Omega)} \le C(h^\ell + \|u - u^h\|^2_{H^1(\Omega)}), \text{ for all } h \le h_1.$$

Substituting (8) into (9), we obtain $\|u - u^h\|_{H^1(\Omega)} \to 0$ for $h \to 0$. We deduce in the above inequality $1 - C\|u - u^h\|_{H^1(\Omega)} \ge \delta > 0$ for all $h \le h_2$, with h_2 small enough (but independent of the particular solution u^h) hence, (6) is established for all $h \le \min\{h_1, h_2\}$. The estimate (7) is deduced by substituting (6) into (10) with $\mu = 1$. The uniqueness of the FEM solution follows from Theorem 3.2. □

Theorem 3.2 *Consider u^h a solution of (4). Under the assumptions of Theorem 3.1, there exist $h_0, \delta > 0$ such that if $h \le h_0$ and $\sigma_h\|z^h_0 - u^h\|_{H^1(\Omega)} \le \delta$, then the sequence $\{z^h_k\}$ for the Newton method* [3]

$$N_h(z^h_k; z^h_{k+1} - z^h_k, v^h) = F_h(v^h) - (a(z^h_k)\nabla z^h_k, \nabla v^h)_h, \quad \forall v^h \in S^\ell_0(\Omega, \mathcal{T}_h), \tag{11}$$

is well defined, and

$$\|z^h_{k+1} - u^h\|_{H^1(\Omega)} \le C\sigma_h\|z^h_k - u^h\|^2_{H^1(\Omega)}, \tag{12}$$

where C is a constant independent of h, k.

In the above theorem, $\sigma_h := \sup_{v^h\in S^\ell_0(\Omega,\mathcal{T}_h)} \|v^h\|_{L^\infty(\Omega)}/\|v^h\|_{H^1(\Omega)}$. Using the quasi-uniformity of the family of triangulations, one can show the standard estimates $\sigma_h \le C(1 + |\ln h|)^{1/2}$ for $d = 2$, and $\sigma_h \le Ch^{-1/2}$ for $d = 3$, where C is independent of h.

3. We define $N_h(z^h; v^h, w^h) := (a(\cdot, z^h)\nabla v^h, \nabla w^h)_h + (v^h \partial_u a(\cdot, z^h)\nabla z^h, \nabla w^h)_h$.

Remark 1 Notice that the requirement of a quasi-uniform mesh for the family of triangulations is often assumed for the analysis of FEM for nonlinear problems [11,16,10]. In our proof, we need it in Step 3 to have an a-priori estimate in $W^{1,6}(\Omega)$ for the FEM solution (with numerical quadrature) associated to L^*. We further need this assumption in the uniqueness result below. However, if $\|u\|_{H^2(\Omega)}$ or the Lipschitz constant are small enough such that $C\lambda^{-1}\Lambda_1\|u\|_{H^2(\Omega)} < 1$, where C depends only on Ω and the polynomial degree of the FE space, then (5) and $u \in W^{1,\infty}(\Omega)$ are not required to prove the uniqueness result, and removing in addition the assumptions of quasi-uniform meshes and $h \leq h_0$, the H^1 estimate (6) still holds.

4. Application to numerical homogenization.

We consider a class of nonlinear nonmonotone multiscale problems

$$-\nabla \cdot (a^\varepsilon(x, u_\varepsilon(x)))\nabla u_\varepsilon(x)) = f(x) \ \text{ in } \Omega, \qquad u_\varepsilon(x) = 0 \ \text{ on } \partial\Omega, \tag{13}$$

with a $d \times d$ tensor $a^\varepsilon(x, x)$ satisfying (2), (3) uniformly in ε. Here ε represent a small scale in the problem. The following homogenization result is shown in [6, Theorem 3.6]: there exists a subsequence of $\{a^\epsilon(\cdot, s)\}$ (again indexed by ε) such that the corresponding sequence of solutions $\{u_\varepsilon\}$ converges weakly to u_0 in $H^1(\Omega)$, where u_0 is solution of the so-called homogenized problem

$$-\nabla \cdot \big(a^0(x, u_0(x))\nabla u_0(x)\big) = f(x) \ \text{ in } \Omega, \qquad u_0(x) = 0 \ \text{ on } \partial\Omega, \tag{14}$$

with a homogenized tensor $a^0(x, s)$ which can be shown to have similar properties as assumed for $a^\varepsilon(x, s)$.

The FE-HMM method for computing a numerical approximation u^H of u_0, essentially similar to the method proposed in [10] [4] reads as follows. It is based on a macroscopic FEM defined on QF with a macro FE space $S_0^\ell(\Omega, \mathcal{T}_H)$ (defined as in Sect. 2), and microscopic FEMs recovering the missing macroscopic tensor at the macroscopic quadrature points. For each macro element $K \in \mathcal{T}_H$ and each integration point $x_{K_j} \in K$, $j = 1, \ldots, J$, we define the sampling domains $K_{\delta_j} = x_{K_j} + (-\delta, \delta)^d$, $(\delta \geq \varepsilon)$. For each K_{δ_j}, we then define a micro FE space $S^q(K_{\delta_j}, \mathcal{T}_h) \subset W(K_{\delta_j})$ with simplicial or quadrilateral FEs and a conformal and shape regular family of triangulations $\mathcal{T}_h$. The space $W(K_{\delta_j})$ is either the Sobolev space $W(K_{\delta_j}) = W^1_{per}(K_{\delta_j}) = \{z \in H^1_{per}(K_{\delta_j});\ \int_{K_{\delta_j}} z dx = 0\}$ for a periodic coupling or $W(K_{\delta_j}) = H^1_0(K_{\delta_j})$ for a coupling through Dirichlet boundary conditions.

FE-HMM Find $u^H \in S_0^\ell(\Omega, \mathcal{T}_H)$ such that $B_H(u^H; u^H, w^H) = F_H(w^H), \forall w^H \in S_0^\ell(\Omega, \mathcal{T}_H)$, where

$$B_H(u^H; v^H, w^H) := \sum_{K\in\mathcal{T}_H} \sum_{j=1}^{J} \frac{\omega_{K_j}}{|K_{\delta_j}|} \int_{K_{\delta_j}} a^\varepsilon(x, u^H(x_{K_j}))\nabla v_{K_j}^{h,u^H(x_{K_j})}(x) \cdot \nabla w_{K_j}^{h,u^H(x_{K_j})}(x) dx, \tag{15}$$

and $w_{K_j}^{h,u^H(x_{K_j})}$ (and similarly for $v_{K_j}^{h,u^H(x_{K_j})}$) denotes the solution of the following micro problem (16) with parameter $s = u^H(x_{K_j})$. Find $w_{K_j}^{h,s}$ such that $w_{K_j}^{h,s} - (w^H(x_{K_j}) + (x - x_{K_j}) \cdot \nabla w^H(x_{K_j})) \in S^q(K_{\delta_j}, \mathcal{T}_h)$ and

$$\int_{K_{\delta_j}} a^\varepsilon(x, s)\nabla w_{K_j}^{h,s}(x) \cdot \nabla z^h(x) dx = 0 \quad \forall z^h \in S^q(K_{\delta_j}, \mathcal{T}_h). \tag{16}$$

We make the following smoothness and structure assumptions on the tensor.

4. In [10] (15) is based on exact micro functions v_{K_j}, w_{K_j} instead of the FE micro functions $v_{K_j}^{h,s}, w_{K_j}^{h,s}$ and the micro-problems are nonlinear (see [10, equs. (5.3)-(5.4)]).

(**H1**) Given $q \in \mathbb{N}$, the cell functions $\psi_{K_j}^{i,s} \in W(K_{\delta_j})$ such that

$$\int_{K_{\delta_j}} a^\varepsilon(x,s)\nabla\psi_{K_j}^{i,s}(x)\cdot\nabla z(x)dx = -\int_{K_{\delta_j}} a^\varepsilon(x,s)\mathbf{e}_i\cdot\nabla z(x)dx,\ \forall z\in W(K_{\delta_j}). \tag{17}$$

satisfy the bound $|\psi_{K_j}^{i,s}|_{H^{q+1}(K_{\delta_j})} \leq C\varepsilon^{-q}\sqrt{|K_{\delta_j}|}$, with C independent of ε, the quadrature point x_{K_j}, the domain K_{δ_j}, and the parameter s for all $i = 1\dots d$. Here, $\mathbf{e}_1,\dots,\mathbf{e}_d$ denotes the canonical basis of $\mathbb{R}^d$. The same assumption also holds with the tensor a^ε replaced by $(a^\varepsilon)^T$ in (17).
(**H2**) for all $m,n = 1,\dots,d$, we assume $a^\varepsilon_{mn}(x,s) = a_{mn}(x,x/\varepsilon,s)$, where $a_{mn}(x,y,s)$ is y-periodic in Y, and the map $(x,s)\mapsto a_{mn}(x,\cdot,s)$ is Lipschitz continuous and bounded from $\overline{\Omega}\times\mathbb{R}$ into $W^{1,\infty}_{per}(Y)$.

Following the framework presented in Sect. 3, we obtain the following H^1 and L^2 a-priori estimates.

Theorem 4.1 *[5] Let $\ell \geq 1$, $q \geq 1$ and $\mu = 0$ or 1. In addition to the assumptions of Theorem 3.1 on problem* (14)*, assume (*$\mathbf{H1}$*), (*$\mathbf{H2}$*), and assume that a^ε satisfies* (2), (3)*. Then, there exist $H_0 > 0$ and $r_0 > 0$ such at if $H \leq H_0$ and $h/\varepsilon \leq r_0$ then*

$$\|u_0 - u^H\|_{H^{1-\mu}(\Omega)} \leq \begin{cases} C(H^{\ell+\mu} + (h/\varepsilon)^{2q} + \delta), & \text{if } W(K_{\delta_j}) = W^1_{per}(K_{\delta_j}) \text{ and } \delta/\varepsilon\in\mathbb{N},\\ C(H^{\ell+\mu} + (h/\varepsilon)^{2q}), & \text{if } W(K_{\delta_j}) = W^1_{per}(K_{\delta_j}) \text{ and } \frac{\delta}{\varepsilon}\in\mathbb{N}, \text{ and } a^\varepsilon(x,s) \\ & \text{is replaced by } a(x_{K_j},x/\varepsilon,s) \text{ in } (15),(16),(17),\\ C(H^{\ell+\mu} + (h/\varepsilon)^{2q} + \delta + \varepsilon/\delta), & \text{if } W(K_{\delta_j}) = H^1_0(K_{\delta_j})\ (\delta > \varepsilon),\end{cases}$$

where we also assume $\delta \leq r_0$ or $\delta + \varepsilon/\delta \leq r_0$ in the first and third cases, respectively. We use the notation $H^0(\Omega) = L^2(\Omega)$. The constants C are independent of $H, h, \varepsilon, \delta$.
If in addition to the assumptions of Theorem 4.1, the map $s\in\mathbb{R}\mapsto a^\varepsilon(\cdot,s)\in(W^{1,\infty}(\Omega))^d$ is of class C^2 with first and second derivatives bounded by $C\varepsilon^{-1}$, then for sufficiently fine meshes and modeling errors (e.g. in the second case of Theorem 4.1, for $(h/\varepsilon)^{2q} \leq H \leq H_1$), one can show the convergence of a Newton method, and the uniqueness of the numerical solution u^H. Notice that in the third case with non-perodic boundary conditions for the micro-macro coupling (i.e. $W(K_{\delta_j}) = H^1_0(K_{\delta_j})$) we obtain the resonance error estimate $r_{MOD} \leq C(\delta + \varepsilon/\delta)$ (similar as for linear problems), whereas $r_{MOD} \leq C(\delta + (\varepsilon/\delta)^{1/2})$ has been obtain in [10, Thm. 5.5].

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