Germund Dahlquist's classical papers on Stability Theory

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"You know, I am a multistep man ... and don't tell anybody, but the first program I wrote for the first Swedish computer was a Runge-Kutta code ..."

(G. Dahlquist 1982, after 10 glasses of wine)

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(G. Dahlquist 1982, after 10 glasses of wine)

"Mr. Dahlquist, when is the spring coming ?"
"Tomorrow, at two o'clock."

(Weather forecast, Stockholm 1955)

1. First Dahlquist Barrier (1956, 1959).

"This work must certainly be considered as one of the great classics in numerical analysis"

(Å. Björk, C.-E. Fröberg 1985).

CONVERGENCE AND STABILITY IN THE NUMERICAL INTEGRATION OF ORDINARY DIFFERENTIAL EQUATIONS

GERMUND DAHLQUIST

1. Introduction and summary

1.1. Statement of the problem. Consider a class of difference equations

$$\alpha_k y_{n+k} + \alpha_{k-1} y_{n+k-1} + \dots + \alpha_0 y_n = h(\beta_k f_{n+k} + \dots + \beta_0 f_n), \tag{1.1}$$

1.2. A numerical example. Apply the formula

$$y_{n+2} = -4y_{n+1} + 5y_n + h(4f_{n+1} + 2f_n)$$

n	Case I (numerical solution)		Case IIa (numerical solution)		Case IIb $\zeta_1{}^n$ with six correct dec.	
	y_n	$10^6 \cdot$ error	y_n	$10^6 \cdot$ error		$10^6 \cdot$ error
0	1,000000	0	1,000000	0	1	
1	1,105171	0	1,105168	3	1,10516781	3
2	1,221384	19	1,221395	8	1,221396	7
3	1,349907	−48	1,349852	7	1,349847	12
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6	1,815963	6156	1,821623	496	1,822088	31
7	2,042538	−28785	2,015902	−2149	2,013713	40
8	2,089871	135670	2,215192	10349	2,225491	50
9	3,097662	−638059	2,507999	−48396	2,459541	62
10	−0,284254	3,002536	2,490202	228080	2,718205	77

Theorem 4a. *The degree p of a stable operator of order k can never exceed $k+2$. If an operator is stable, then the condition that $R(z)$ is an odd function is necessary and sufficient for the degree to be equal to $k+2$. All roots of $R(z)$ are then located on the imaginary axis. If k is odd, the degree of a stable operator cannot exceed $k+1$.*

Theorem 4b. *If an operator of even order k is stable, then the conditions*

$$\alpha_\nu = -\alpha_{k-\nu}, \qquad \beta_\nu = \beta_{k-\nu} \tag{2.22}$$

are necessary and sufficient in order that it should be of maximum degree $k+2$. All roots of $\varrho(\zeta)$ then have unit modulus.

"The main result is rather negative (Thm. 4), but there are new formulas of this general class which are at least comparable ,."

(G. Dahlquist 1956.)

Proof.

$$\varrho(\zeta) \equiv \alpha_k \zeta^k + \alpha_{k-1}\zeta^{k-1} + \dots + \alpha_0 ,$$
$$\sigma(\zeta) \equiv \beta_k \zeta^k + \beta_{k-1}\zeta^{k-1} + \dots + \beta_0 .$$

$$\zeta = (z+1)/(z-1), \qquad z = (\zeta+1)/(\zeta-1),$$
$$R(z) = \left(\tfrac{1}{2}(z-1)\right)^k \varrho(\zeta) \equiv \sum_{j=0}^{k} a_j z^j ,$$
$$S(z) = \left(\tfrac{1}{2}(z-1)\right)^k \sigma(\zeta) \equiv \sum_{j=0}^{k} b_j z^j .$$

In these notations the relation (2.15) transforms into

$$R(z) - S(z)\log\frac{z+1}{z-1} \sim -C\left(\frac{2}{z}\right)^{p-k+1} \qquad (z\to\infty), \tag{2.17}$$

$$R(z)\left(\log\frac{z+1}{z-1}\right)^{-1} - S(z) \sim -C\left(\frac{2}{z}\right)^{p-k} \qquad (z\to\infty). \tag{2.18}$$

$$\left(\log \frac{z+1}{z-1}\right)^{-1} = \frac{z}{2} - \sum_{\nu=0}^{\infty} \mu_{2\nu+1}\, z^{-(2\nu+1)}$$

$$\begin{aligned}
\mu_{2\nu+1} &= -\frac{1}{2\pi i} \int_C z^{2\nu} \left(\log \frac{z+1}{z-1}\right)^{-1} dz \\
&= -\frac{1}{2\pi i} \int_{-1}^{1} x^{2\nu} \left(\pi^2 + \log^2 \frac{1+x}{1-x}\right)^{-1} \cdot \\
&\qquad \cdot \left(\left(-\pi i + \log \frac{1+x}{1-x}\right) - \left(\pi i + \log \frac{1+x}{1-x}\right)\right) dx \\
&= \int_{-1}^{1} x^{2\nu} \left(\pi^2 + \log^2 \frac{1+x}{1-x}\right)^{-1} dx > 0\,.
\end{aligned}$$

Thirty years later ...

ROYAL INSTITUTE OF TECHNOLOGY
DEPARTMENT OF NUMERICAL ANALYSIS
AND COMPUTING SCIENCE
NADA
S-100 44 STOCKHOLM 70, SWEDEN
TEL: 08-787 7000

March 31, 1987

Professor Gerhard Wanner
Université de Genève
Section de Mathematiques
2-4 rue de Lièvre

Dear Gerhard,

My sincerest thanks for your excellent book that is really a hall-mark in the literature on ODE's. It has everything: an excellent choice of topics, clarity and elegance in the presentation, a better presentation of comparative tests and other numerical experiments, than we have ever seen before. And on the top of this: a wonderful humour. I am really pleased by being involved in a few of these humorous remarks.

A couple of weeks after your book, Butcher's book appeared. It is like when you serve yourself ketchup. For a long time nothing comes, and then it comes all at the same time. For me, who knows all the authors, it is great to see how related subjects are treated in such different ways, which reflect your personalities.

Congratulations to this achievement, and I really look forward to seeing "the stiff book".

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... and what can this "modern" ketchup book do better ..?

Instead of

1.2. A numerical example. Apply the formula

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n	CASE I (numerical solution)		CASE IIa (numerical solution)		CASE IIb ζ_1^n with six correct dec.	
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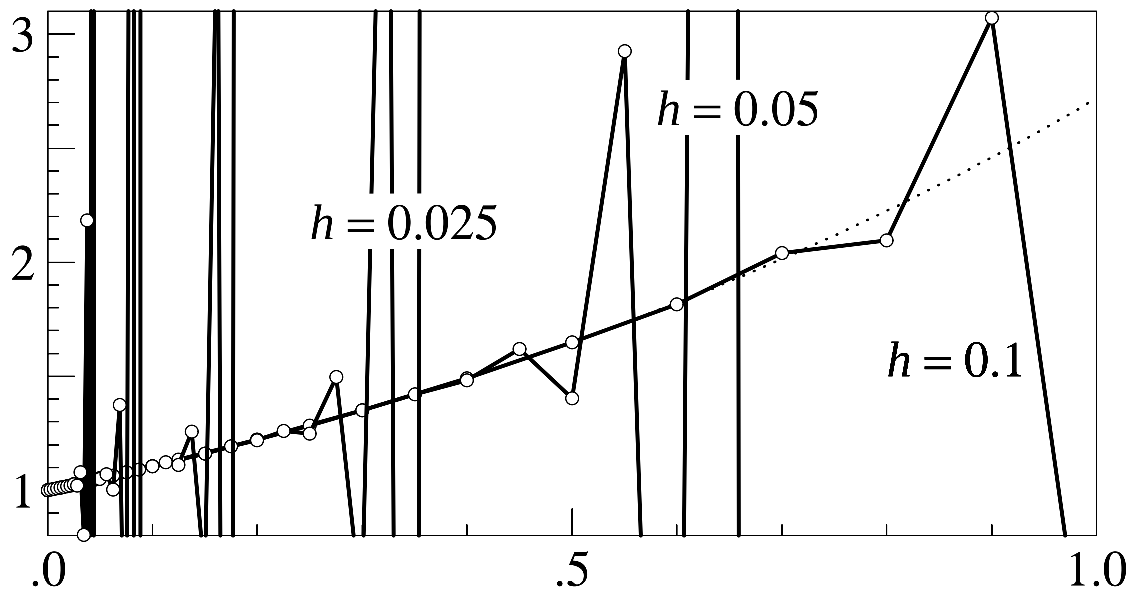
it has

$$y_{n+2} + 4y_{n+1} - 5y_n = h(4f_{n+1} + 2f_n).$$



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Instead of

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it has

$$\zeta = \frac{z+1}{z-1} \qquad \text{or} \qquad z = \frac{\zeta+1}{\zeta-1}$$

$$R(z) = (\frac{z-1}{2})^k \rho(\zeta) = \sum_{j=0}^{k} a_j z^j,$$

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it has

$$R(z)(\log \frac{z+1}{z-1})^{-1} - S(z) = C_{p+1}(\frac{2}{z})^{p-k} + \mathcal{O}((\frac{2}{z})^{p-k+1}) \qquad \text{for } z \to$$

Instead of

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it has

$$(\log \frac{z+1}{z-1})^{-1} = \frac{z}{2} - \mu_1 z^{-1} - \mu_3 z^{-3} - \mu_5 z^{-5} - \dots$$

and finally, instead of

$$\mu_{2\nu+1} = -\frac{1}{2\pi i}\int_C z^{2\nu}\left(\log\frac{z+1}{z-1}\right)^{-1} dz$$

$$= -\frac{1}{2\pi i}\int_{-1}^{1} x^{2\nu}\left(\pi^2 + \log^2\frac{1+x}{1-x}\right)^{-1} \cdot$$

$$\cdot\left(\left(-\pi i + \log\frac{1+x}{1-x}\right) - \left(\pi i + \log\frac{1+x}{1-x}\right)\right) dx$$

$$= \int_{-1}^{1} x^{2\nu}\left(\pi^2 + \log^2\frac{1+x}{1-x}\right)^{-1} dx > 0\ .$$

we read

$$\mu_{2j+1} = -\frac{1}{2\pi i}\int_{-1}^{1} x^{2j}[(\log\frac{1+x}{1-x} + i\pi)^{-1} - (\log\frac{1+x}{1-x} - i\pi)^{-1}]d$$

$$= \int_{-1}^{1} x^{2j}[(\log\frac{1+x}{1-x})^2 + \pi^2]^{-1} dx > 0.$$

...and one can do nothing better ...

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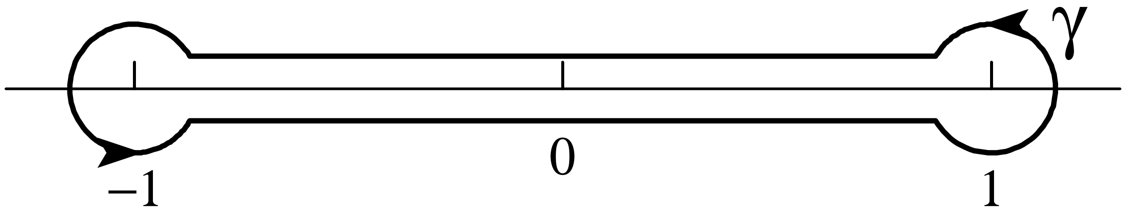
$$= \int_{-1}^{1} x^{2j}[(\log\frac{1+x}{1-x})^2 + \pi^2]^{-1}dx > 0.$$

...and one can do nothing better ... just add a nice picture ...



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$$\mu_{2j+1} = -\frac{1}{2\pi i}\int_{-1}^{1} x^{2j}\Big[\Big(\log\frac{1+x}{1-x}+i\pi\Big)^{-1} - \Big(\log\frac{1+x}{1-x}-i\pi\Big)^{-1}\Big]d$$

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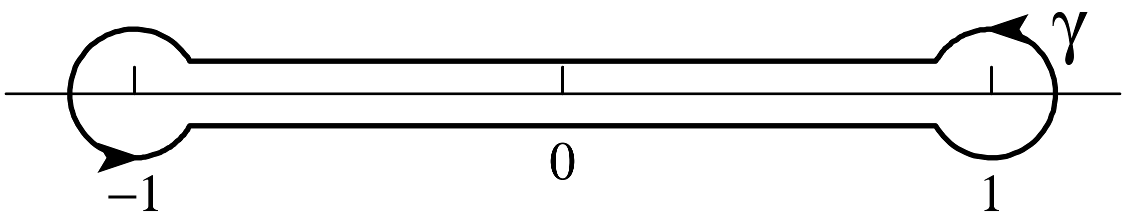


“Although there exist many different proofs for the theorem the original published proof still appears very elegant,...”

(R. Jeltsch, O. Nevanlinna 1985)

2. The Second Dahlquist Barrier (1963).

A SPECIAL STABILITY PROBLEM FOR LINEAR MULTISTEP METHODS*

GERMUND G. DAHLQUIST

Abstract.

The trapezoidal formula has the smallest truncation error among all linear multistep methods with a certain stability property. For this method error bounds are derived which are valid under rather general conditions. In order to make sure that the error remains bounded as $t \to \infty$, even though the product of the Lipschitz constant and the step-size is quite large, one needs not to assume much more than that the integral curve is uniformly asymptotically stable in the sense of Liapunov.

I didn't like all these "strong", "perfect", "absolute", "generalized", "super", "hyper", "complete" and so on in mathematical definitions, I wanted something neutral; and having been impressed by David Young's "property *A*", I chose the term "*A*-stable".

(G. Dahlquist, in 1979).

the famous definition ...

DEFINITION. *A k-step method is called A-stable, if all solutions of* (1.2) *tend to zero, as* $n \to \infty$, *when the method is applied with fixed positive h to any differential equation of the form,*

$$dx/dt = qx\,, \tag{1.8}$$

where q is a complex constant with negative real part.

... and the famous theorem

THEOREM 2.2. *The order, p, of an A-stable linear multistep method cannot exceed 2. The smallest error constant, $c^* = \frac{1}{12}$, is obtained for the trapezoidal rule, $k = 1$, with the generating polynomials* (2.2).

... and the famous theorem

THEOREM 2.2. *The order, p, of an A-stable linear multistep method cannot exceed* 2. *The smallest error constant, $c^* = \frac{1}{12}$, is obtained for the trapezoidal rule, $k=1$, with the generating polynomials* (2.2).

... and some years later ...

“Talking on stiff differential equations in Sweden, is like carrying coals to Newcastle...”

(W.L. Miranker, Göteborg 1975).

“certainly one of the most influential papers ever published in BIT”

(Å. Björk, C.-E. Fröberg 1985).

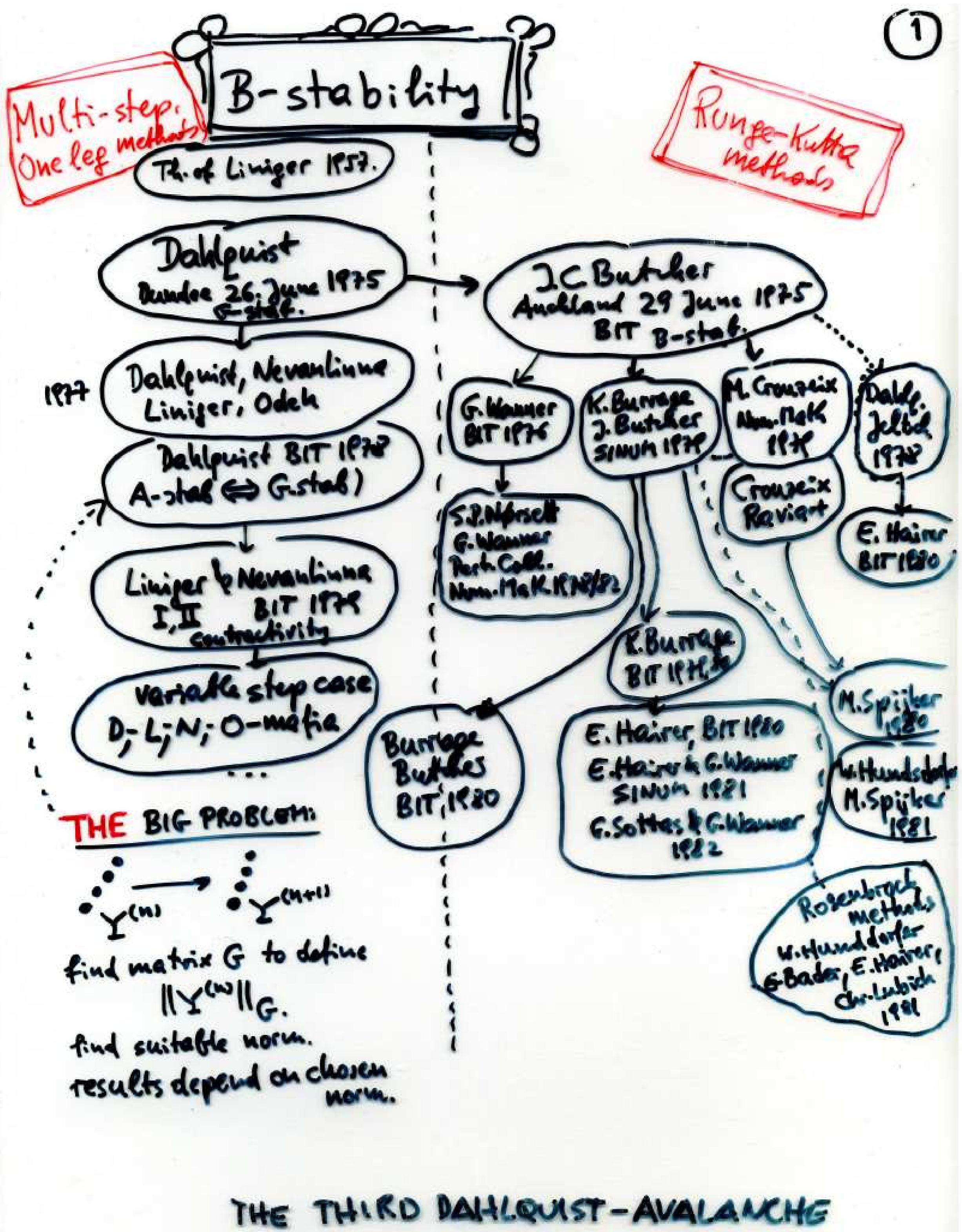
The second ketchup

STABILITÄTSTHEORIEN
FÜR STEIFE DIFFERENTIALGLEICHUNGEN

INHALT:

A-stable
A(α)-stable
A(0)-stable
A_0-stable
Å-stable
AN-stable
A_D-stable
algebraically stable
B-stable, B-consistent, B-convergent
BN-stable
BS-stable
BSI-stable ← C-stable
circle contractive
D-stable
D-stable
G-stable (A-contr.)
I-stable
internally stable
L-stable
internally L-stable
multiplyers
O-stable
P-stable, stable, R-stable, S-stable, stiffly stable, variable Schritte, splitting Methoden, W-methoden ...

The third ketchup



Proofs of Dahlquist's Theorem.

"I searched for a long time, finally Professor Lax showed me the Riesz-Herglotz theorem and I knew that I had my theorem.."

(G. Dahlquist, Stockholm 1979 , private comm.)

analytic functions. Following a suggestion of Professor P. D. Lax (oral communication), we shall use a variant of Riesz–Herglotz' theorem, cf.

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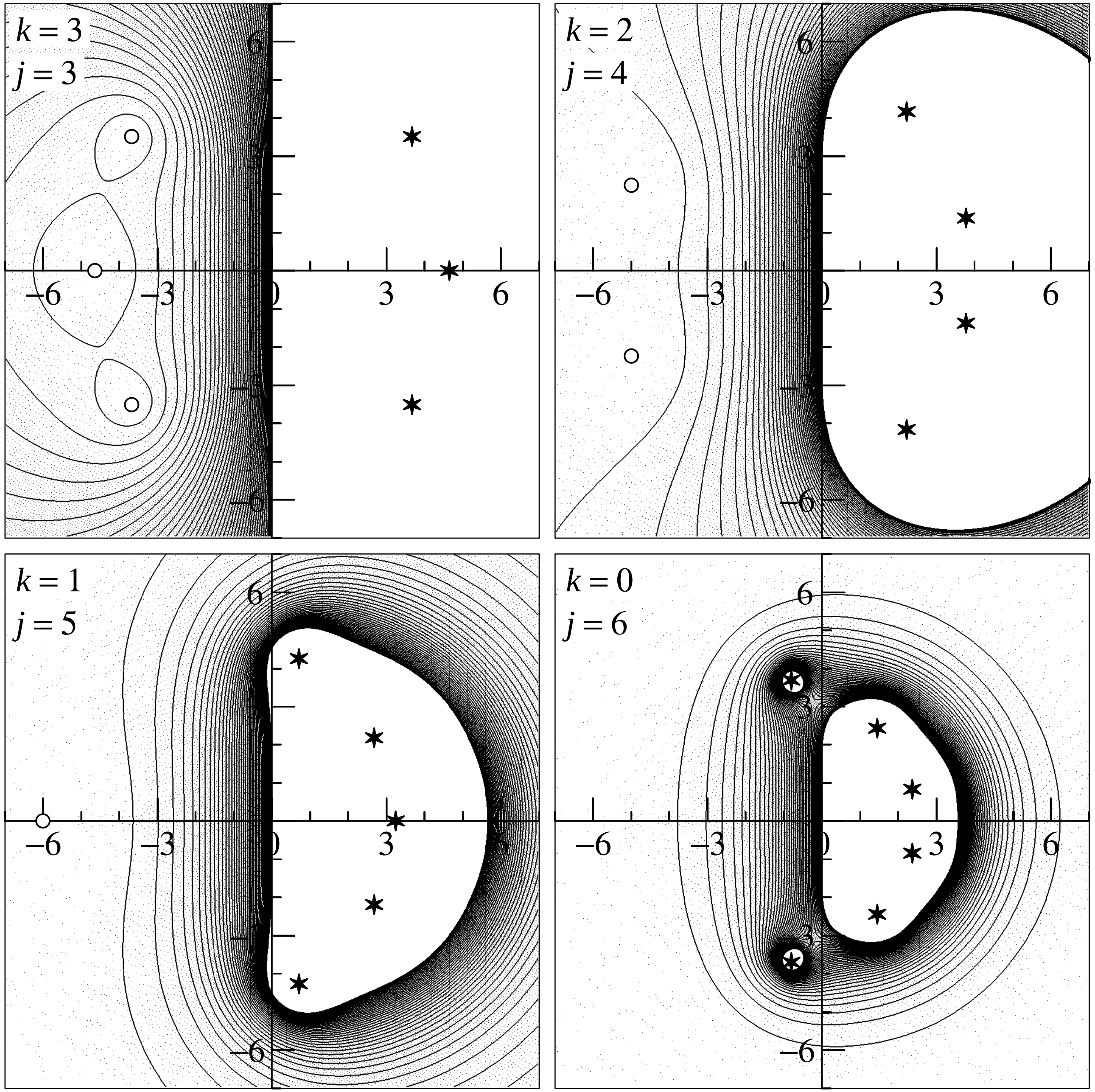
"The stars were, however, not reached until 1978".

(another citation from another preface of another special issue of BIT (vol. 41, No. 5, 2001))

Sincerely yours Gerhard

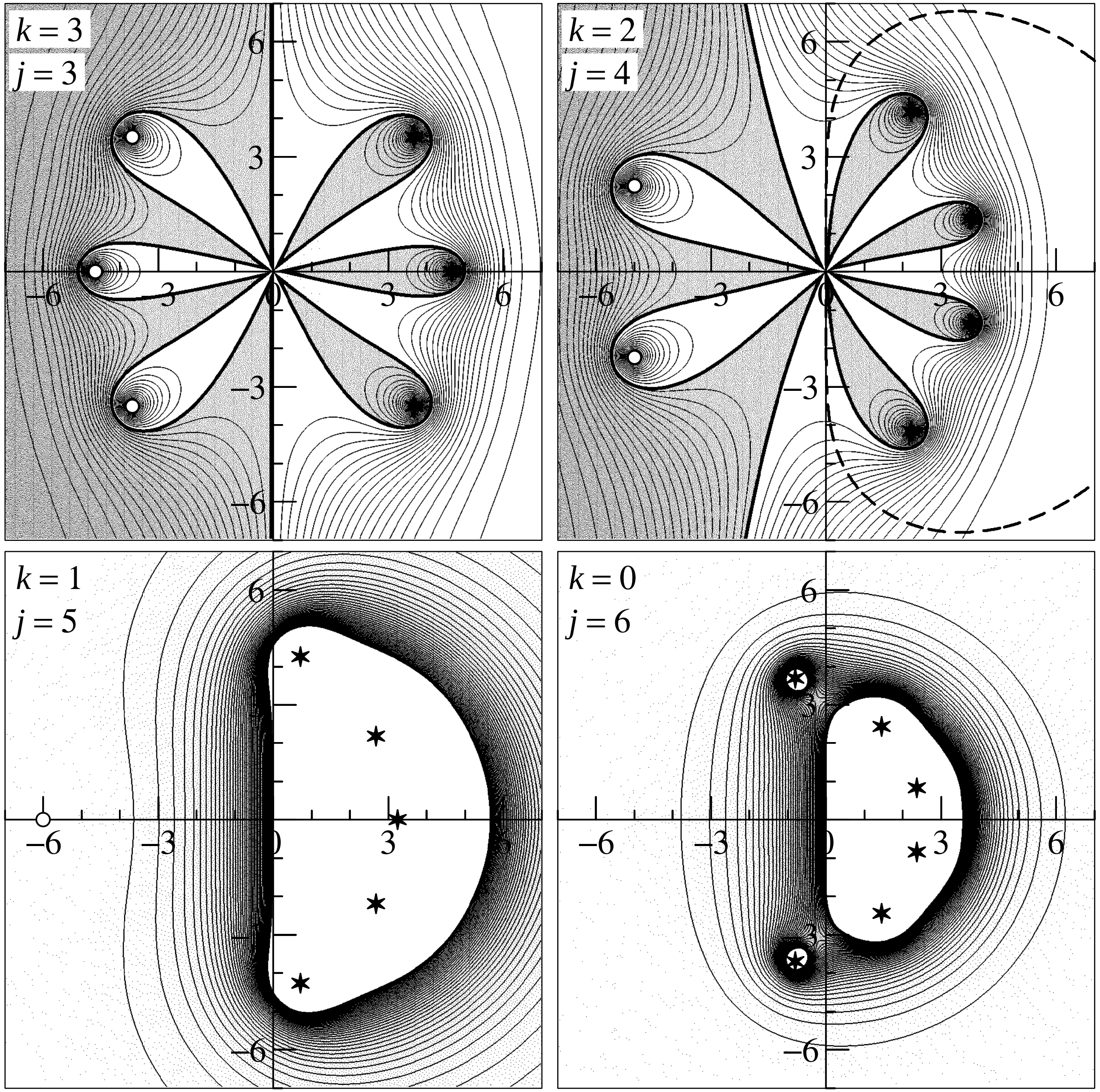
LONG LIVE THE ORDER STARS !!!

Original motiv.: Ehle's Conj. (with E.Hairer and S.Nørsett)



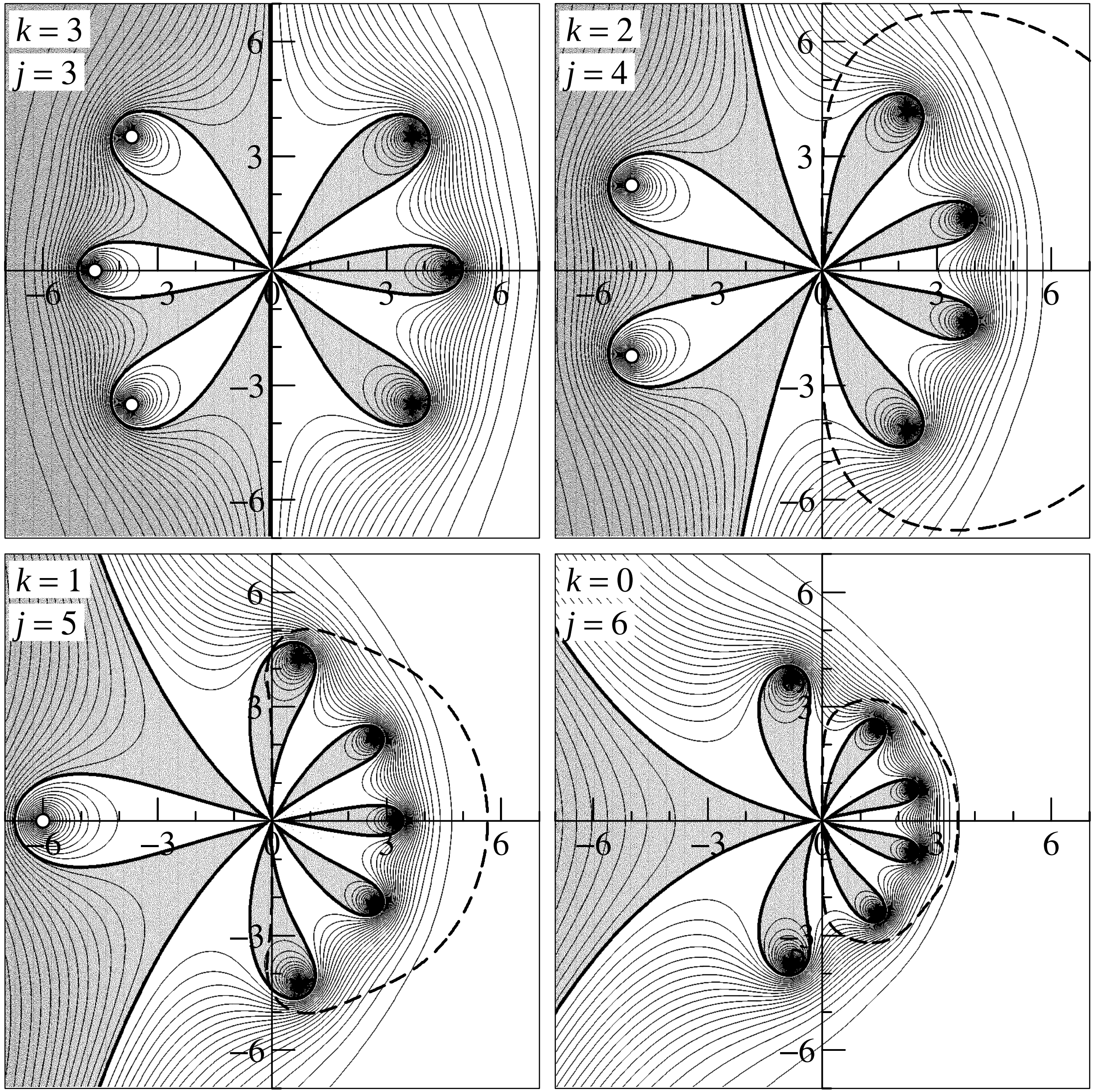
Conjecture. A-stable $\Leftrightarrow$ $k \leq j \leq k+2$.

Ehle's Conjecture ; Order Stars



Theorem. A-stable $\Leftarrow$ $k \leq j \leq k+2$.

Ehle's Conjecture ; Order Stars



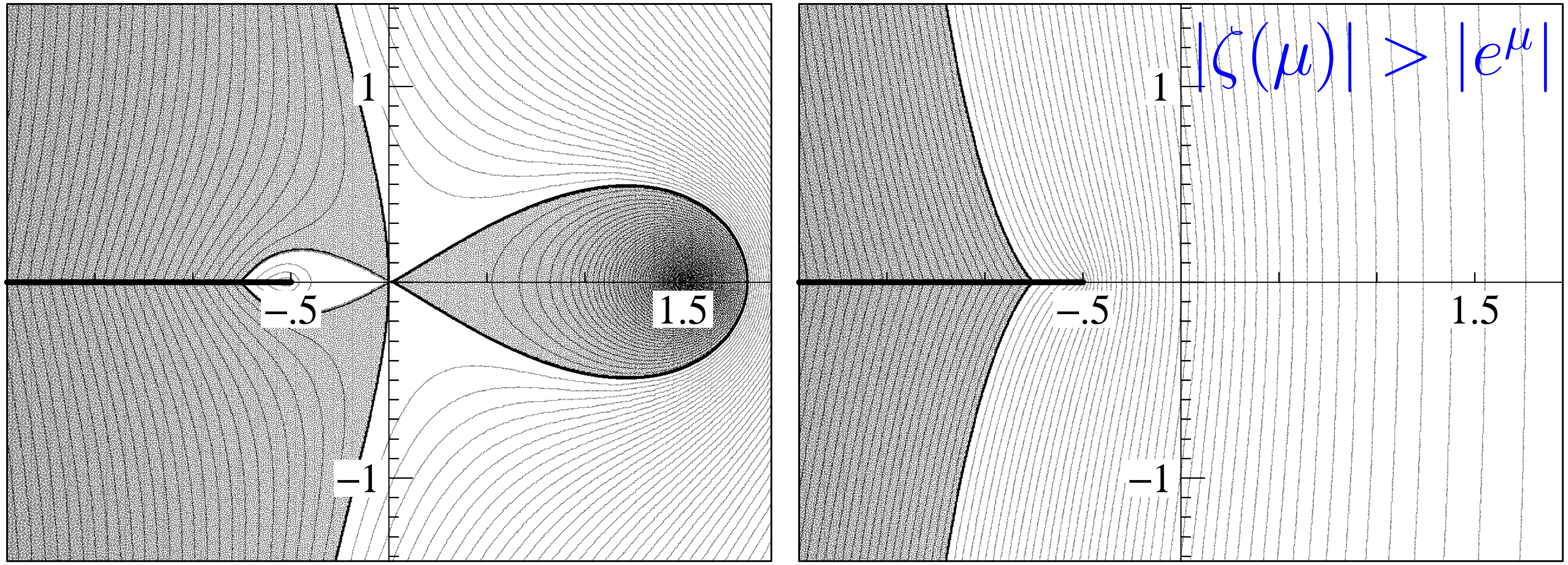
Theorem. A-stable $\Leftrightarrow$ $k \leq j \leq k+2$.

Example: BDF2.

$$\frac{3}{2}y_{n+1} - 2y_n + \frac{1}{2}y_{n-1} = hf_{n+1}$$

$$y' = \lambda y, \quad \mu = h\lambda \quad \Rightarrow \quad (\tfrac{3}{2} - \mu)\zeta^2 - 2\zeta + \tfrac{1}{2} = 0.$$

Algebraic equation for ζ $\quad \Rightarrow \quad \zeta_{1,2}(\mu) = \dfrac{2 \pm \sqrt{1 + 2\mu}}{3 - 2\mu}$



Example: BDF2. • Implicit stage $\Rightarrow$ Pole of ζ

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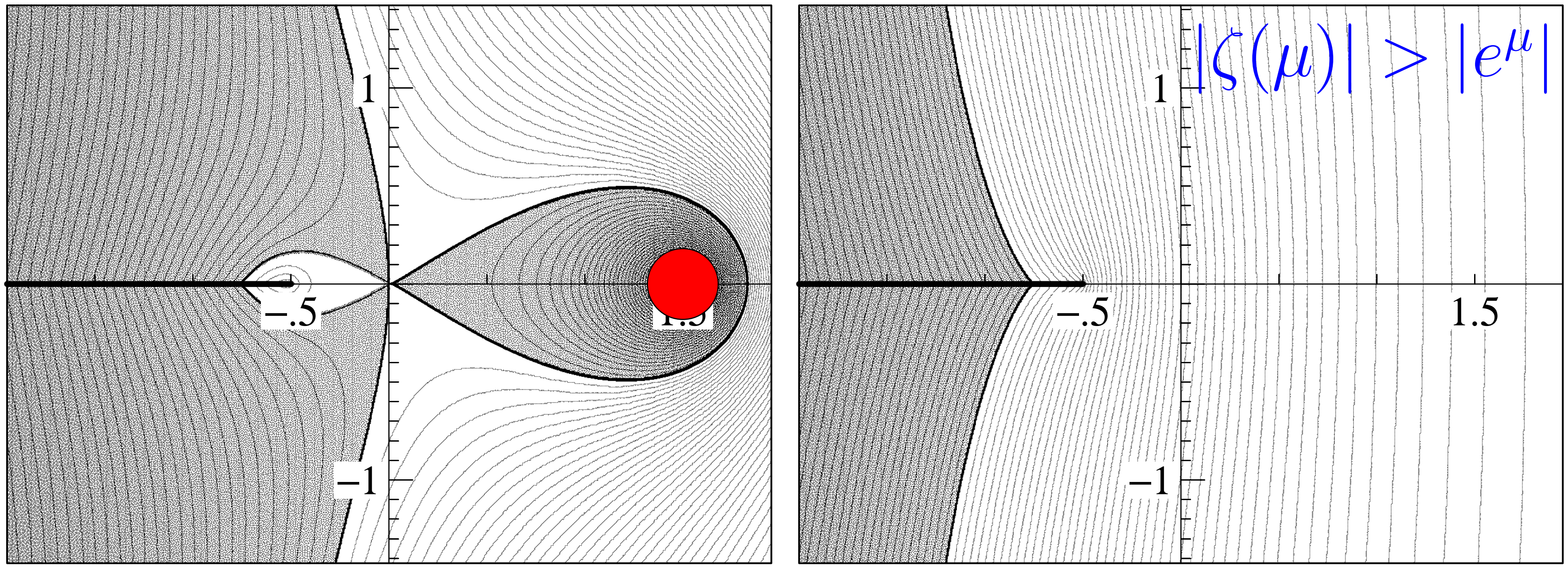

Example: BDF2.

- Implicit stage $\Rightarrow$ Pole of ζ
- Order $\Rightarrow$ $e^{\mu} - \zeta_1(\mu) = C \cdot \mu^3 + \ldots$

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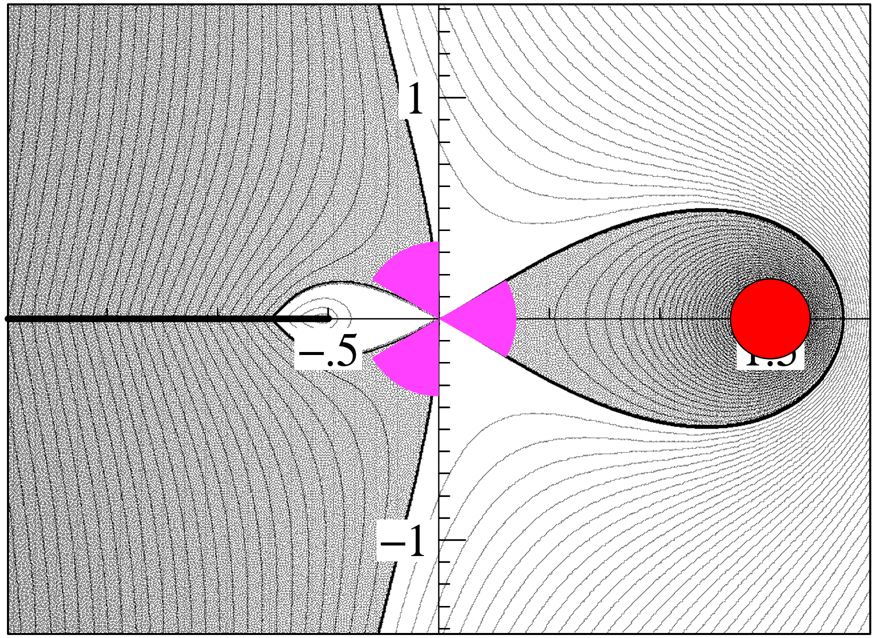
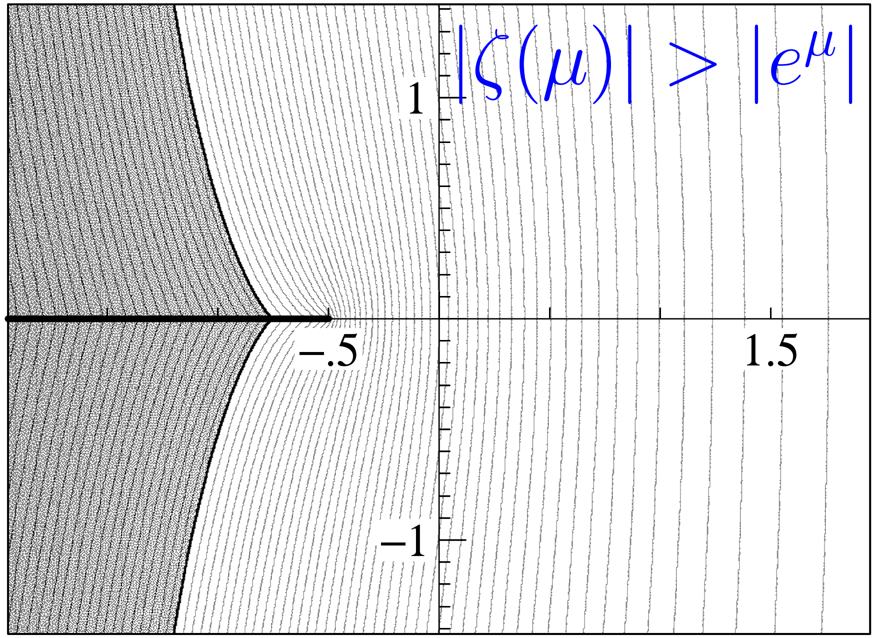
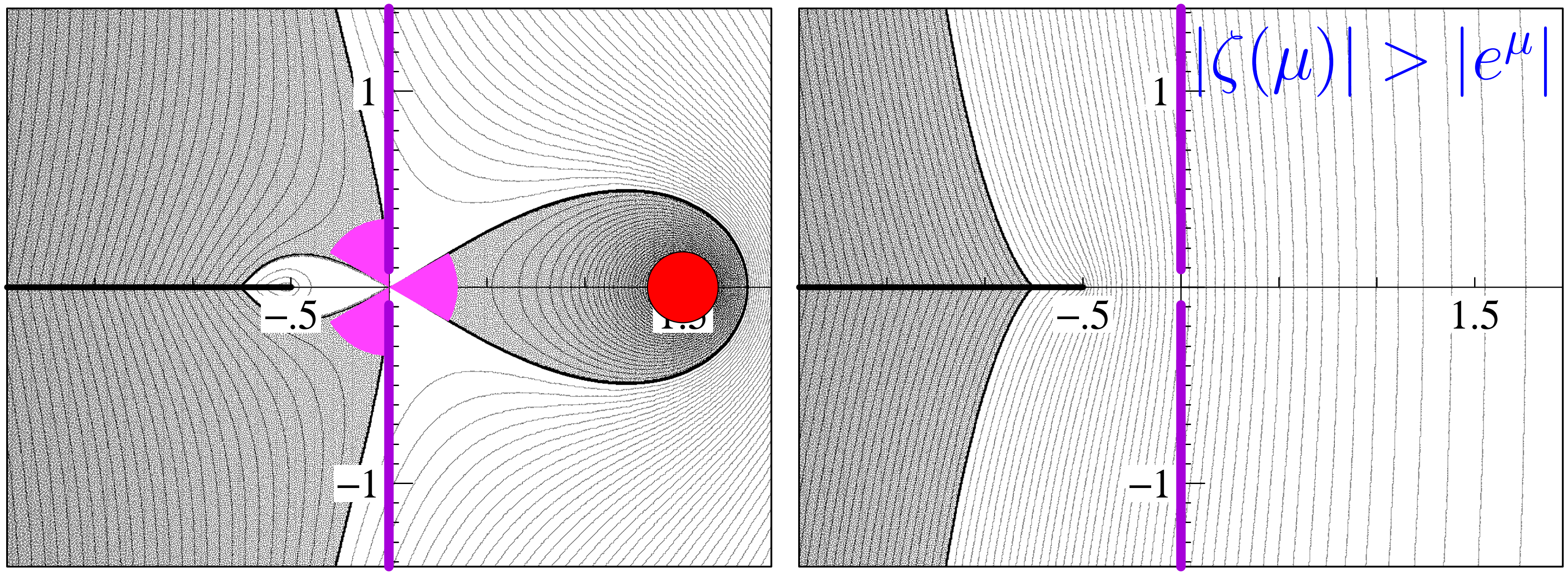




Example: BDF2.

- Implicit stage $\Rightarrow$ Pole of ζ
- Order $\Rightarrow$ $e^{\mu} - \zeta_1(\mu) = C \cdot \mu^3 + \ldots$
- A-stable $\Rightarrow$ order star away from imag. axis.

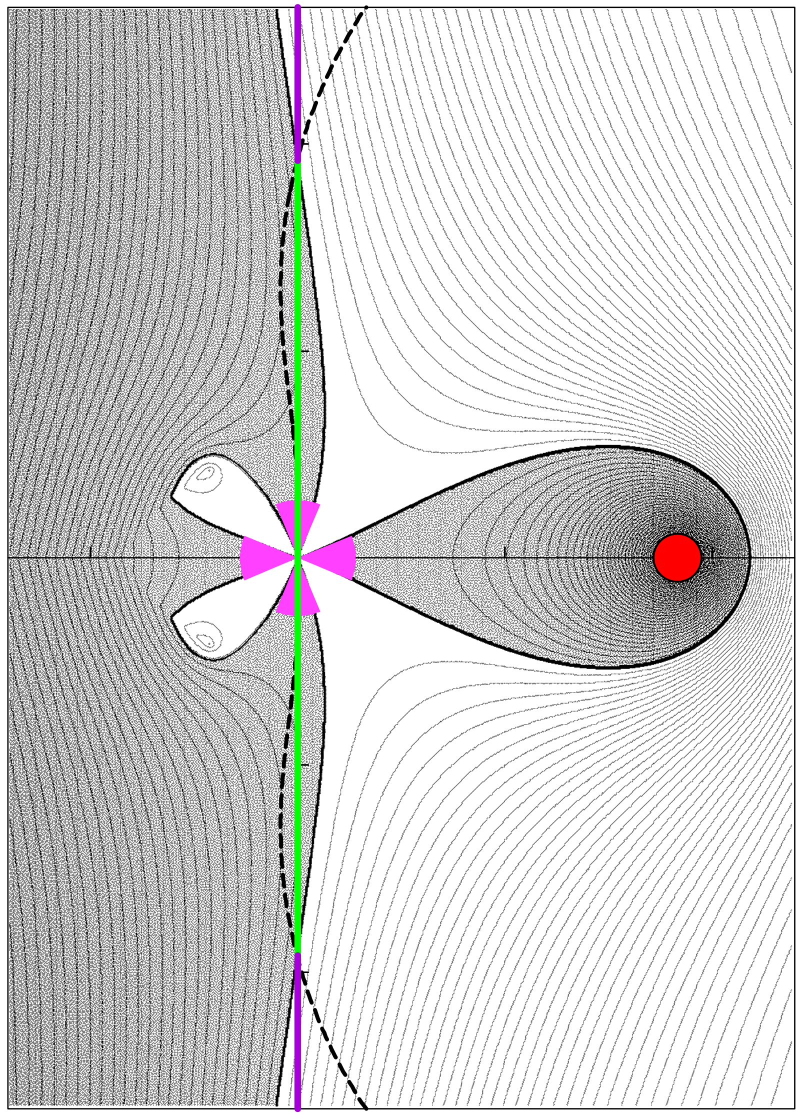


Example: BDF3.

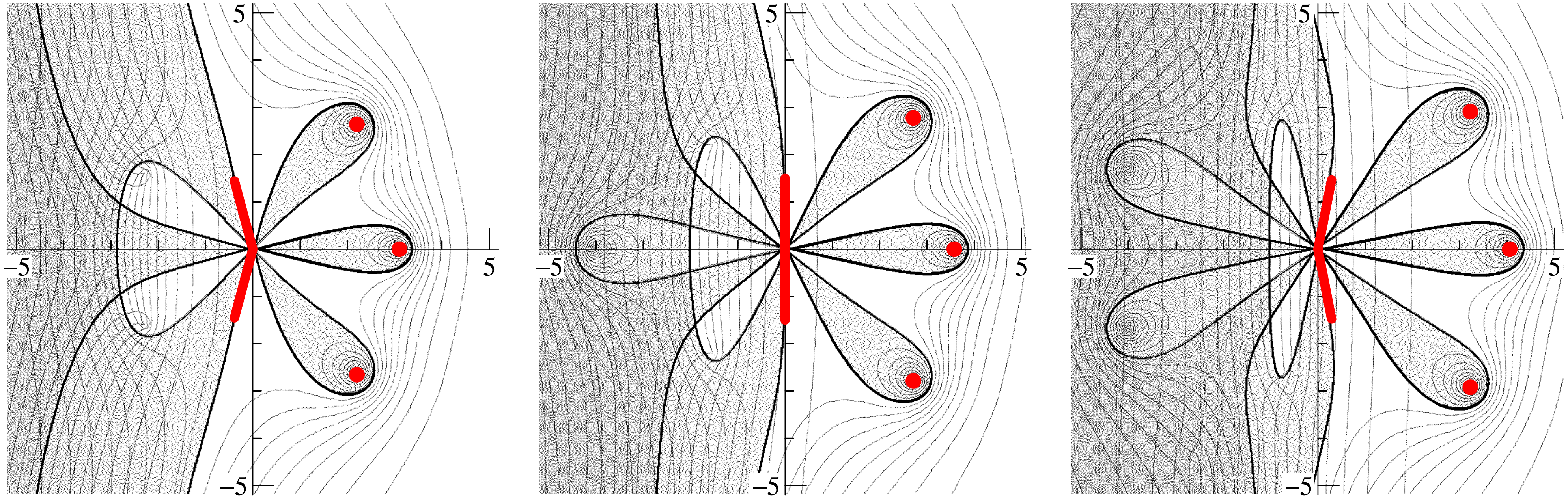
$$\frac{11}{6}y_{n+1} - 3y_n + \frac{3}{2}y_{n-1} - \frac{1}{3}y_{n-2}$$

$$= hf_{n+1}$$

$$(\frac{11}{6} - \mu)\zeta^3 - 3\zeta^2 + \frac{3}{2}\zeta - \frac{1}{3} = 0.$$

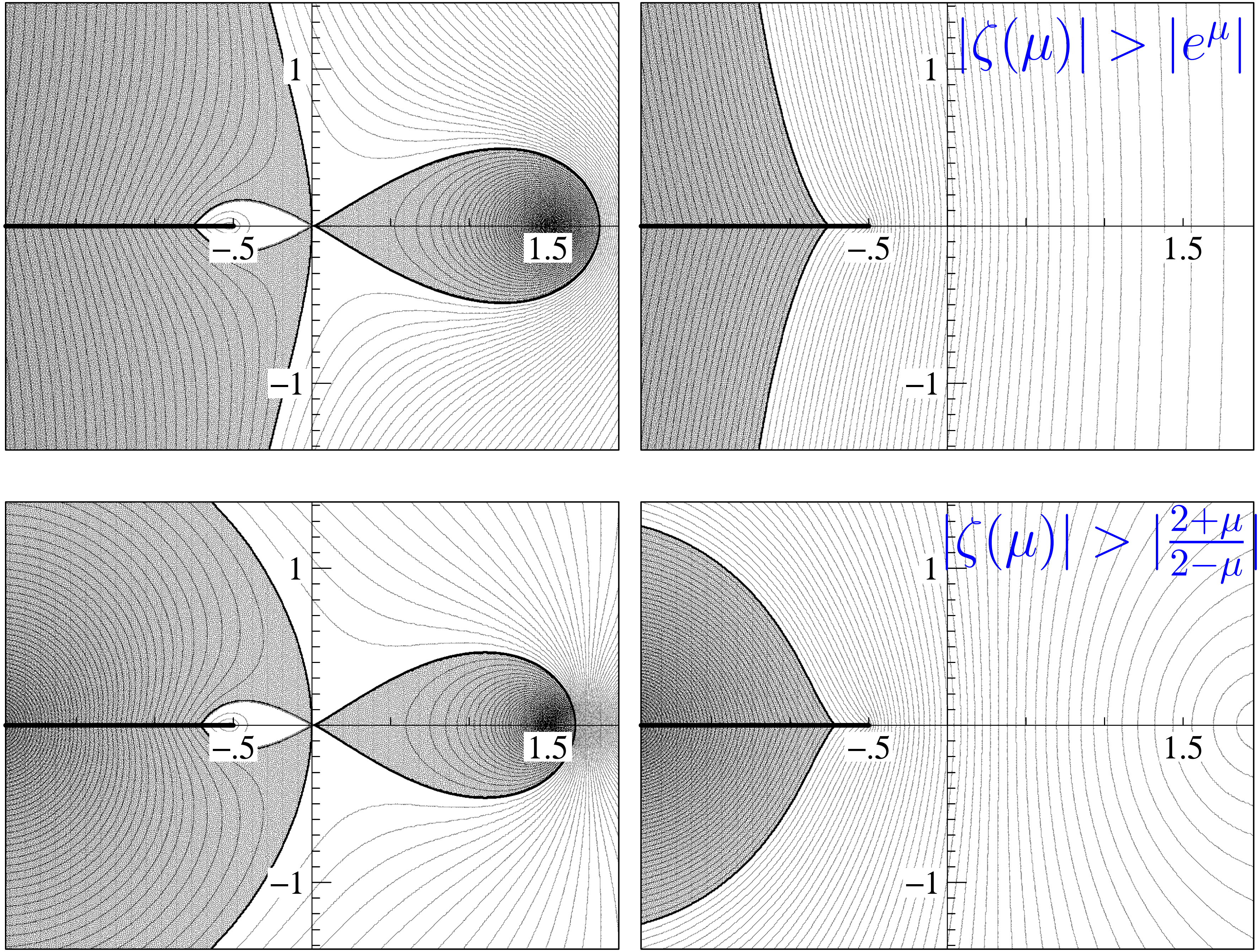


The Daniel-Moore Conjecture.

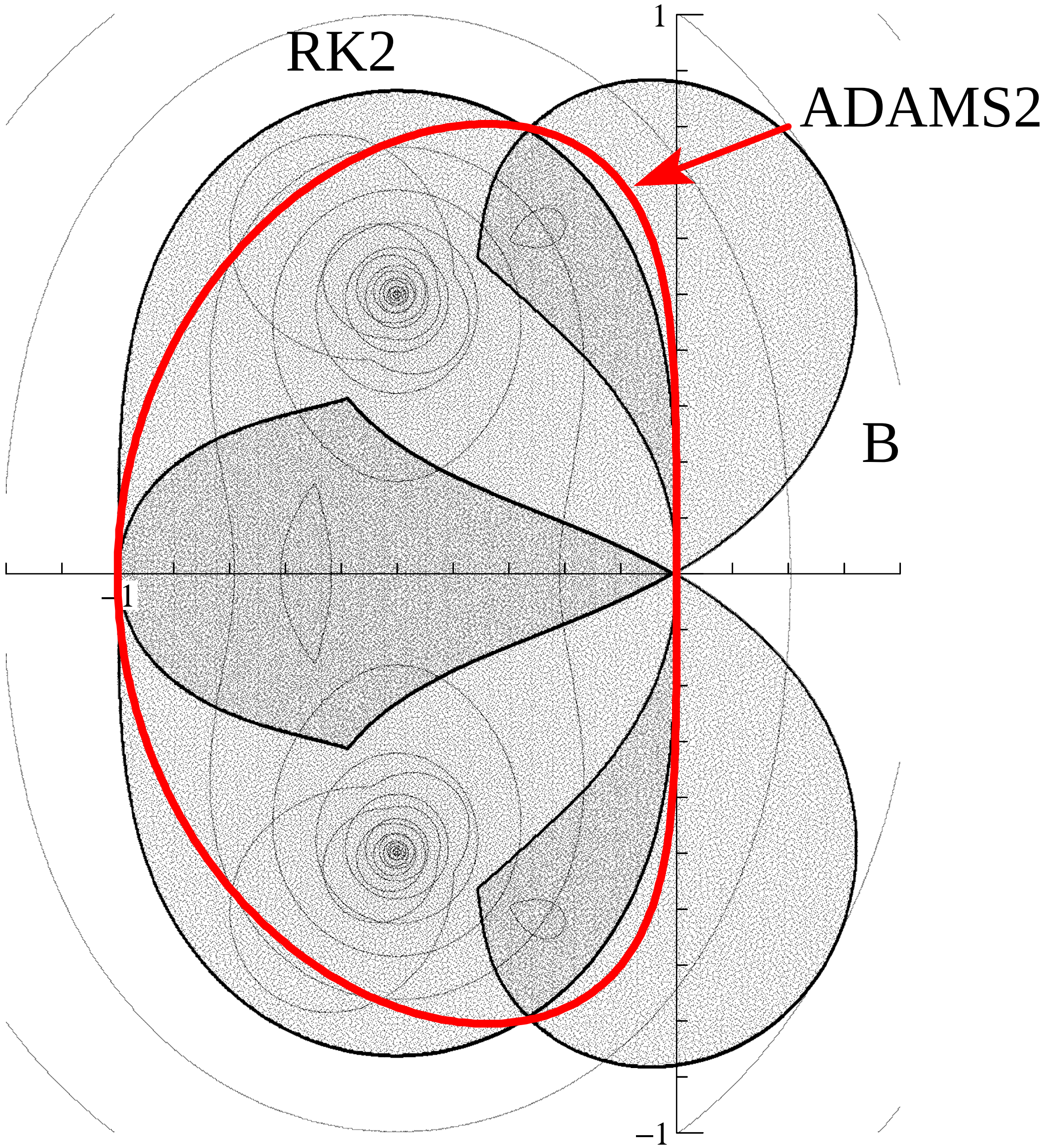


$$p \leq 2s$$

Error constant.



Jeltsch-Nevanlinna Theorem.



$$S_1^{scal} \not\supset S_2^{scal} \qquad \text{and} \qquad S_1^{scal} \not\subset S_2^{scal}$$

Dear Gerhard and Ernst!

My sincerest thanks for the second volume of your Opus Magnum that is a really impressive piece of work, and a documentation of profound thoughts! It covers practically all aspects of the numerical