# Leçon d’adieu

Le petit Larousse illustré, éd. 2008:

**LEÇON** n.f. (lat. *lectio, -onis,* lecture). **1.** Enseignement donné en une séance par un professeur, un maître, à une classe, à un auditoire, à un élève. **2.** Ce que le maître donne à apprendre. *Réciter sa*

a toujours une connotation ennuyeuse ...

**ADIEU** interj. et n.m. (de *à* et *Dieu*). S'emploie pour saluer qqn que l'on ne reverra pas de longtemps ou que l'on ne reverra plus. *Tout est fini entre nous, adieu ! Des adieux déchirants.* ◇ *Dire adieu à*

a toujours une connotation triste ...

**ça promet ... !**

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# ça promet ... !

Mais les couleurs de la vie reviennent, dès qu'on parle de ...

# Kepler, Newton et l'analyse numérique

**16 oct. 2008**

(Merci à E. Hairer, Chr. Aebi, B. Gisin, P. Henry et A. Ostermann)

(Merci à la Bibliothèque de Genève)

Un grand pédagogue reveille l’attention des auditeurs par ...

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**Devinette:**

$$\boxed{\text{Qu'est-ce ?} \qquad 3.14 \qquad =?}$$

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Réponse : La valeur de $\pi$ dans le manuel officiel des collèges genevois des années 1970.

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**Devinette:**

$$\boxed{\text{Qu'est-ce ?} \qquad 314.00 \qquad =?}$$

Un grand pédagogue reveille l'attention des auditeurs par ...

**Devinette:**

$$\boxed{\text{Qu'est-ce ?} \qquad 3.14 \qquad =?}$$

Réponse : La valeur de $\pi$ dans le manuel officiel des collèges genevois des années 1970.

**Devinette:**

$$\boxed{\text{Qu'est-ce ?} \qquad 314.00 \qquad =?}$$

Réponse : La valeur de $100\pi$ du même manuel officiel des collèges genevois des années 1970.

La nomination d'un prof d'analyse numérique par la faculté en 1973 a donc été une très bonne idée ...!!

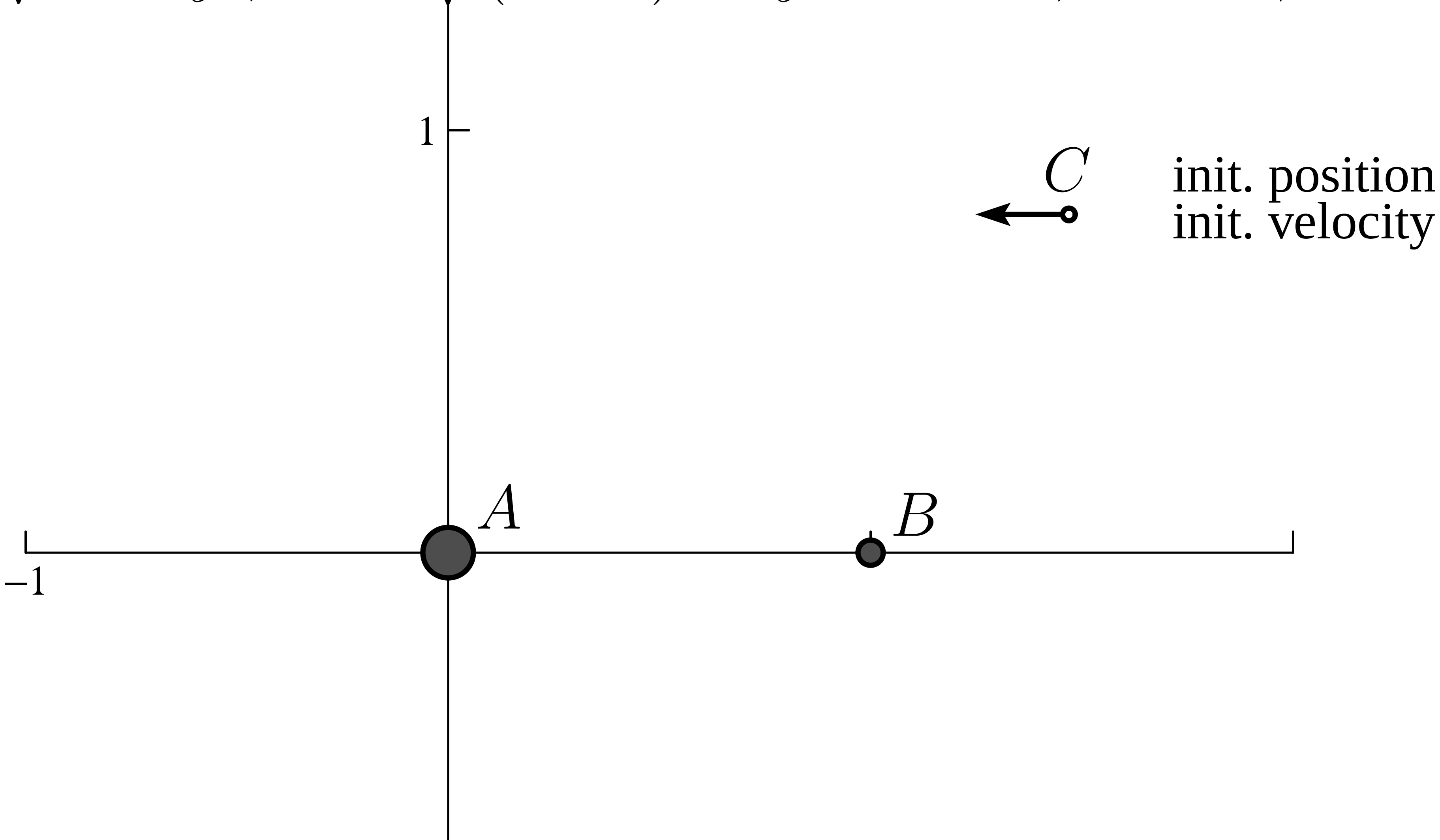
# 1. Numerical Analysis. Always starts with a ...

**Mathematical Problems to Solve**
Integrals, Ordinary differential equations,
PDE’s, Variational problems etc.

# Example: Particle attr. by two fixed centers (Euler E301, 1760)

$$\frac{ddx}{dt^2} = -\frac{Ax}{v^3} - \frac{B(x-a)}{u^3}, \qquad \frac{ddy}{dt^2} = -\frac{Ay}{v^3} - \frac{By}{u^3}$$

$$v = \sqrt{x^2+y^2}, \quad u = \sqrt{(x-a)^2+y^2} \quad A = 2, \quad B = 1, \quad a = 1$$



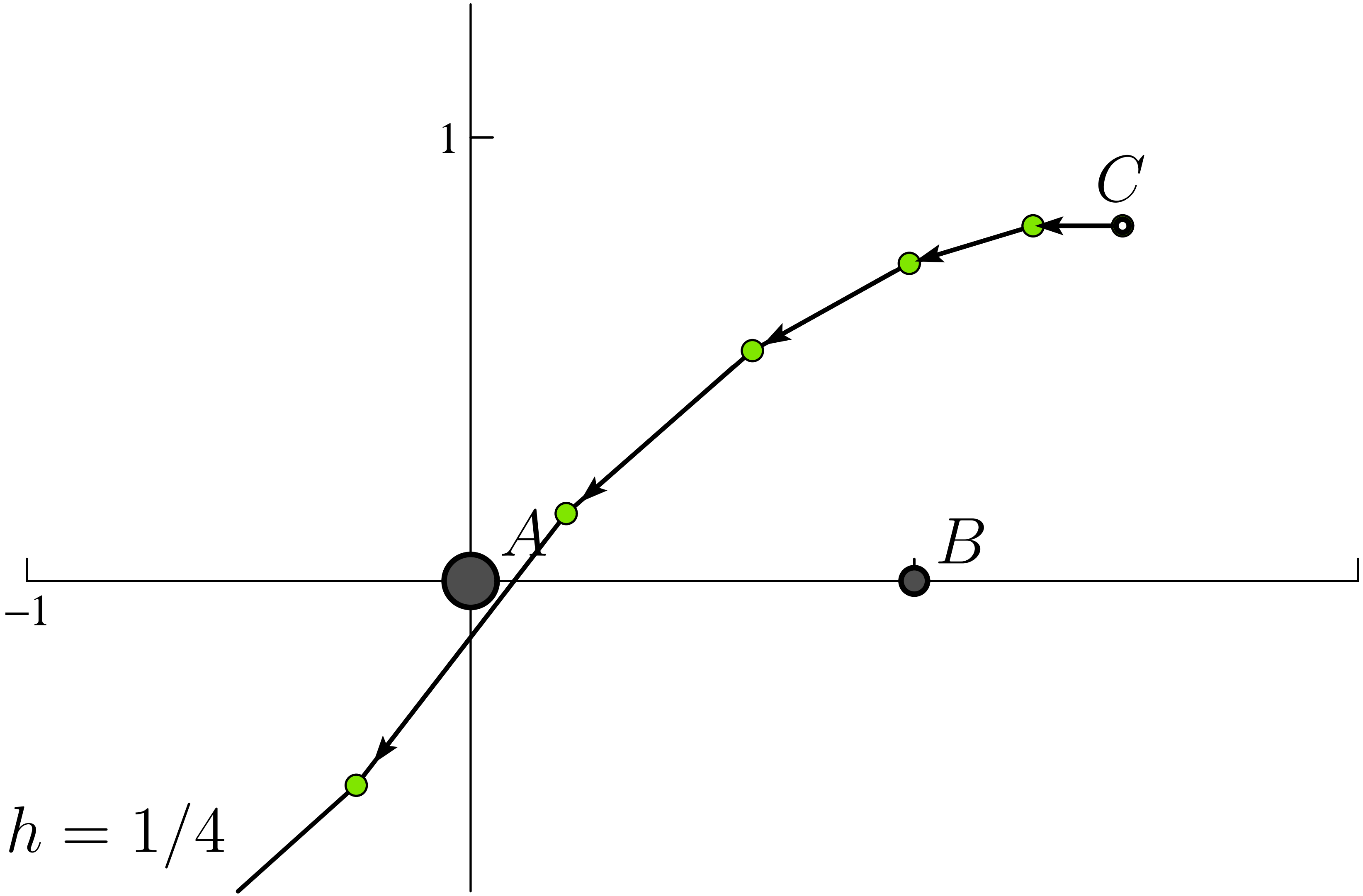
# “Euler’s Method” (E342), Inst. Calc. Integralis 1768, §650:

$$b' = b + A(a' - a)$$

$$b'' = b' + A'(a'' - a')$$

$$b''' = b'' + A''(a''' - a'')$$

| Ipſius | valores ſucceſſiui |
|---|---|
| $x$ | $a\,;\ a'\,;\ a''\,;\ a'''\,;\ a^{IV}\,;\ \ldots\ {}'x\,;\ x$ |
| $y$ | $b\,;\ b'\,;\ b''\,;\ b'''\,;\ b^{IV}\,;\ \ldots\ {}'y\,;\ y$ |
| V | $A\,;\ A'\,;\ A''\,;\ A'''\,;\ A^{IV}\,;\ \ldots\ {}'V\,;\ V.$ |



## “Euler’s Method” (E342), Inst. Calc. Integralis 1768, §650:

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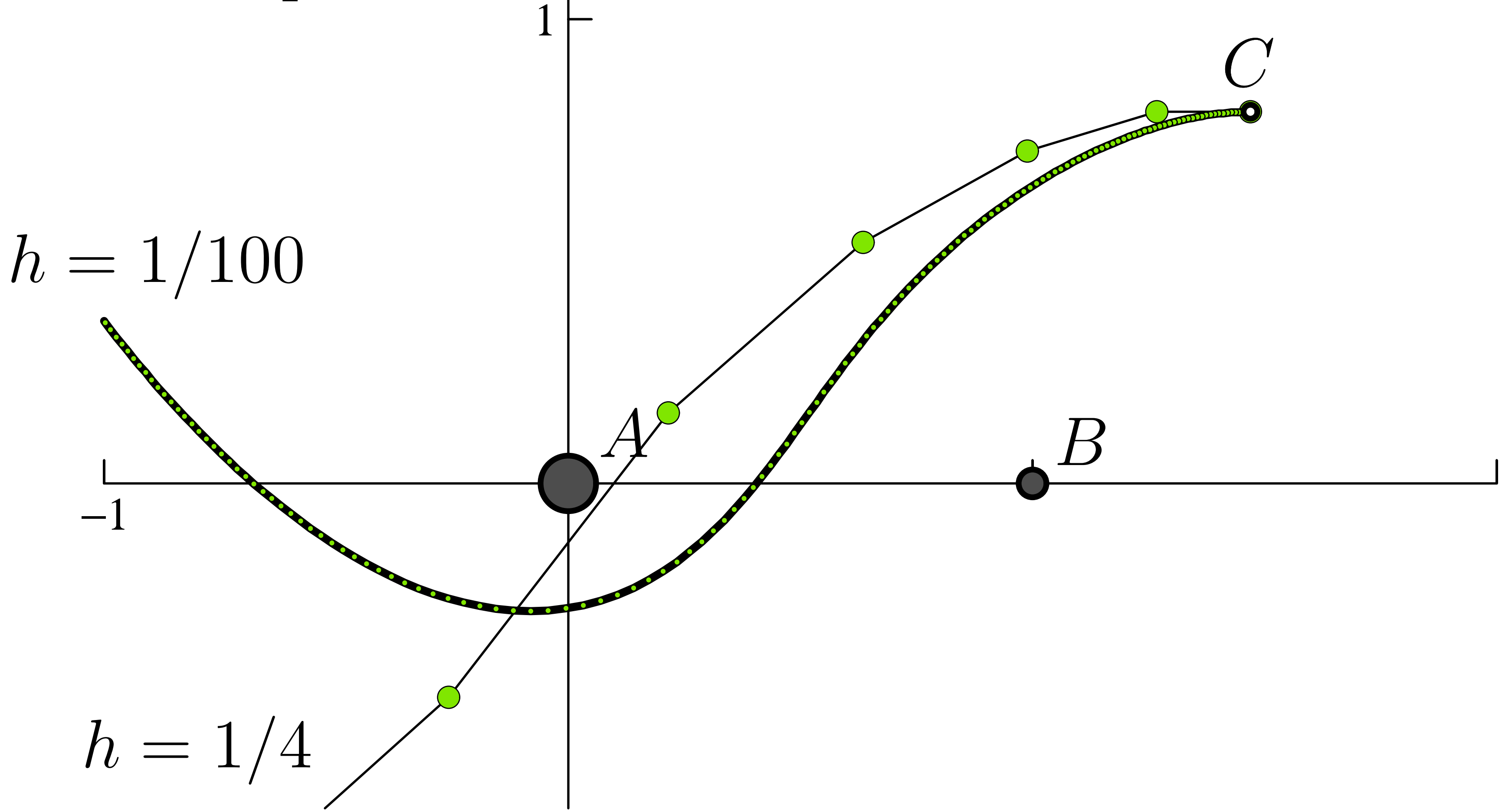
| Ipsius | valores successiui |
|---|---|
| $x$ | $a$ ; $a'$ ; $a''$ ; $a'''$ ; $a^{IV}$ ; . . . . $'x$ ; $x$ |
| $y$ | $b$ ; $b'$ ; $b''$ ; $b'''$ ; $b^{IV}$ ; . . . . $'y$ ; $y$ |
| V | A ; A$'$ ; A$''$ ; A$'''$ ; A$^{IV}$ ; . . . . $'$V ; V. |

Decrease step size $h$:

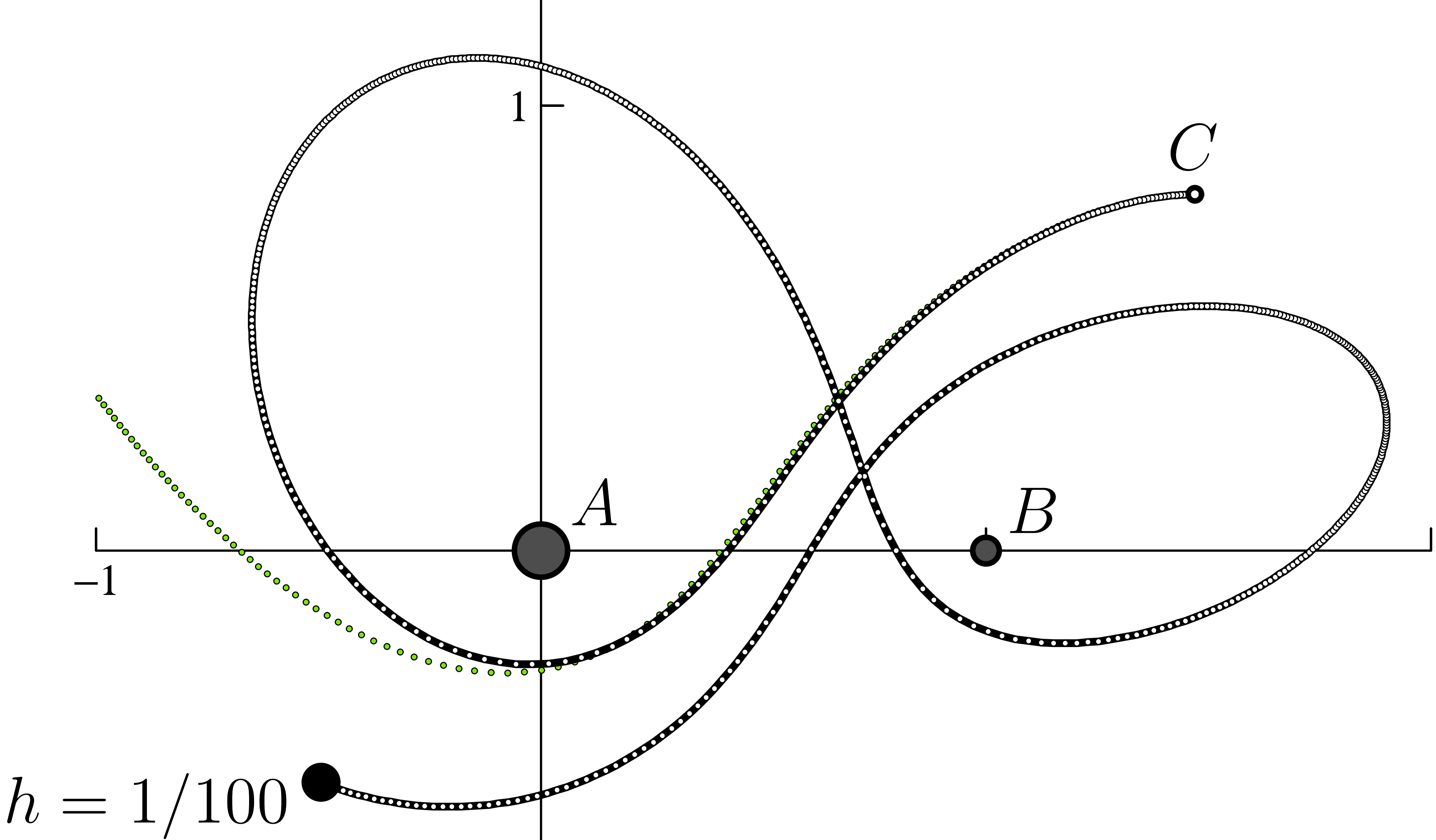


## Second Order Method (Euler E342), ICI 1768, §656:

$$y = b + \frac{(x-a)dy}{dx} - \frac{(x-a)^2 ddy}{1\cdot 2\, dx^2} + \frac{(x-a)^3 d^3y}{1\cdot 2\cdot 3\, dx^3} - \frac{(x-a)^4 d^4y}{1\cdot 2\cdot 3\cdot 4\, dx^4} + \text{etc.}$$

$$\frac{ddy}{dx^2} = \left(\frac{dV}{dx}\right) + V\left(\frac{dV}{dy}\right)$$

$$\frac{d^3y}{dx^3} = \left(\frac{ddV}{dx^2}\right) + \left(\frac{dV}{dx}\right)\left(\frac{dV}{dy}\right) + 2V\left(\frac{ddV}{dxdy}\right) + V\left(\frac{dV}{dy}\right)^2 + VV\left(\frac{ddV}{dy^2}\right).$$



All this is "dirty work" , hence in brown color,...

**"Brute" construction of Numerical Methods**
Quadrature formulas, Runge-Kutta methods, Multistep methods,
FE-, FD-methods, programming, grid-adaption ...

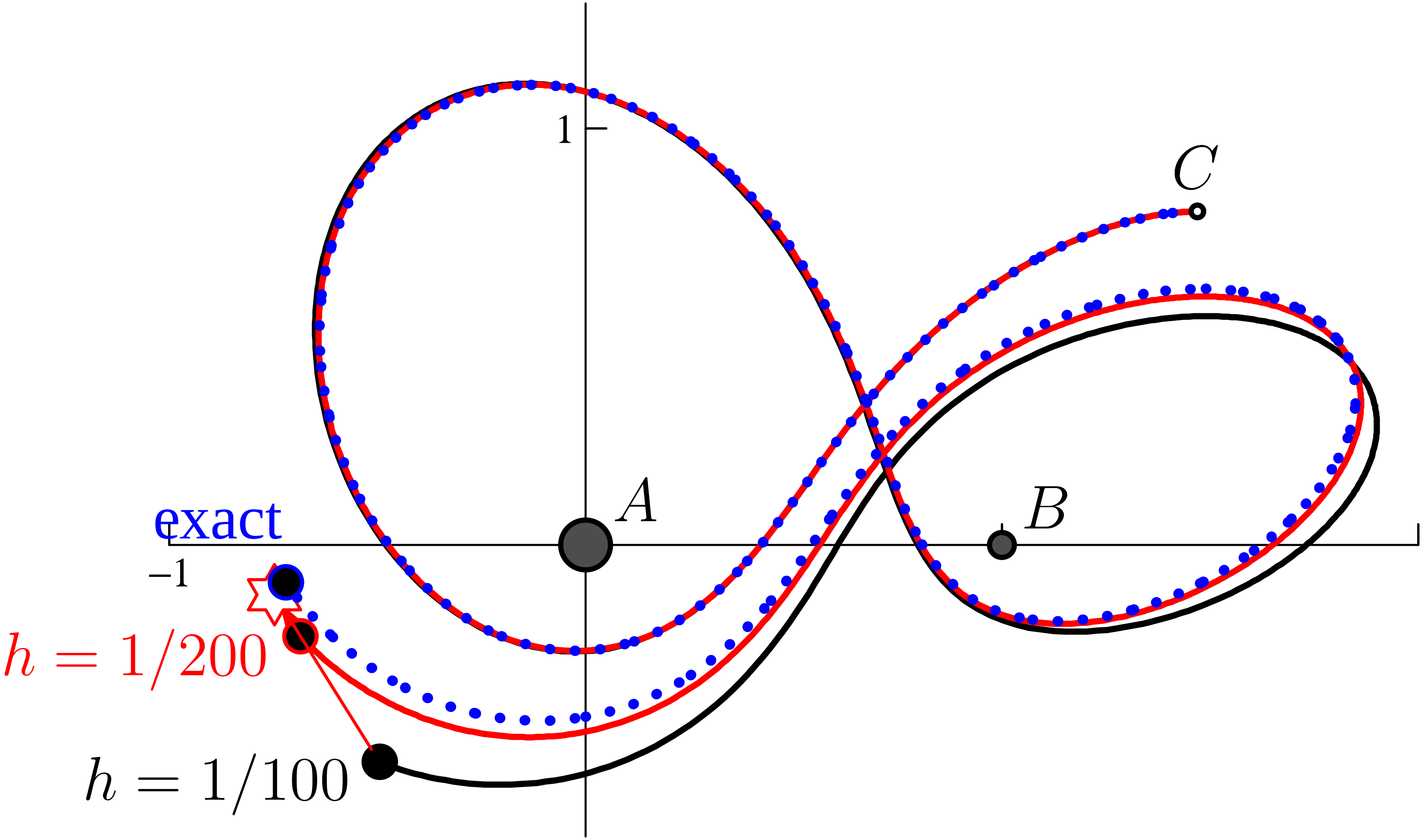
Numerical methods

**Mathematical Problems to Solve**
Integrals, Ordinary differential equations,
PDE's, Variational problems etc.

# Error Estimations ?

(Cauchy 1824, Runge 1894, Vietoris 1953, Henrici 1962 . . .)

Example: Method of order 2: $\boxed{\text{err} \leq C \cdot h^2}$




We arrive at our final picture (blue sky again) . . .

**Theory of Numerical Methods**
Order theory, Stability theory, Contractivity,
Convergence theory, Order Barriers, Order Stars

↑

More respectability

↑

**“Brute” construction of Numerical Methods**
Quadrature formulas, Runge-Kutta methods,
Multistep methods, FE-, FD-methods

↑

Numerical methods

↑

**Mathematical Problems to Solve**
Integrals, Ordinary differential equations,
PDE’s, Variational problems etc.

## Example 1:

**On the Butcher Group and General Multi-Value Methods**

By

**E. Hairer**, Innsbruck, and **G. Wanner**, Genève

With 7 Figures

Received October 29, 1973

$$B(a,y) = a(\emptyset)y + h\,a(\bullet)f(y) + h^2 a([\bullet])f'(y)f(y) + \dots$$

$$\begin{tikzcd} y_0 \arrow[r, bend left, "B(a,y_0)"] \arrow[rr, bend right, "B(a,y_0)"'] & y_1 \arrow[r, bend left, "B(a,y_1)"] & y_2 \end{tikzcd}$$

**Theorem.** If $a(\emptyset) = 1$, then $\boxed{B(a, B(a,y)) = B(a,y)}$ where

$$a(\bullet) = a(\emptyset)a(\bullet) + a(\bullet)$$

$$a([\bullet]) = a(\emptyset)a([\bullet]) + a(\bullet)a(\bullet) + a([\bullet])$$

$$a([\bullet,\bullet]) = a(\emptyset)a([\bullet,\bullet]) + a(\bullet)a(\bullet)^2 + 2a([\bullet])a(\bullet) + a([\bullet,\bullet])$$

# Example 2:

# Order Stars (1978)

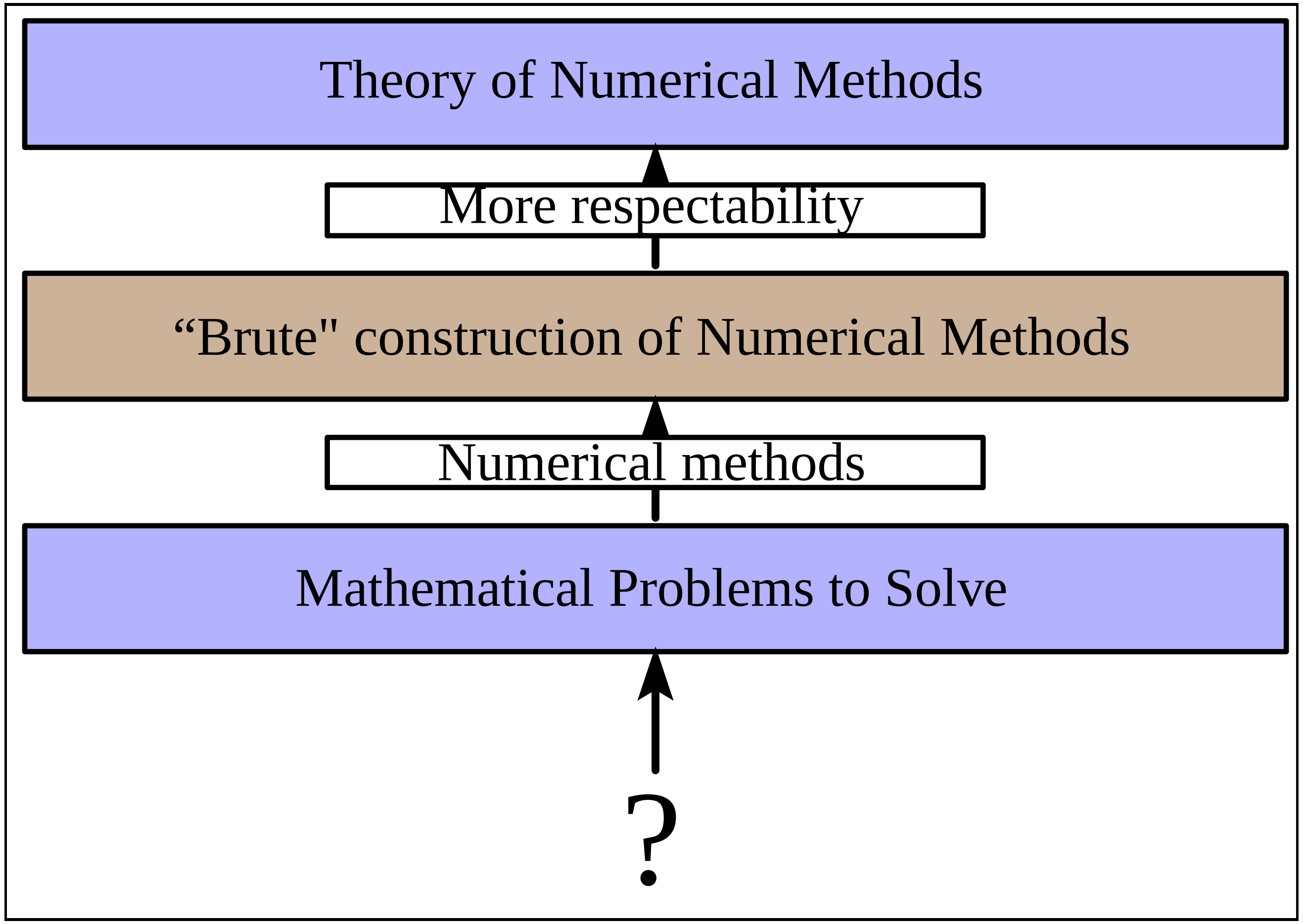
Stability theorems
and order bounds
for stiff differential equations

Prof. L. N. Trefethen — Phone: (607) 255-4222
Dept. of Computer Science — Fax: " 255-4428
Upson Hall, Cornell University
Ithaca, NY 14853-7501 — Email: lnt@cs.cornell.edu

"CLASSIC PAPERS IN NUMERICAL ANALYSIS"

| | |
|---|---|
| 1. Cooley & Tukey (1965) | the Fast Fourier Transform |
| 2. Courant, Friedrichs & Lewy (1928) | finite difference methods for PDE |
| 3. Householder (1958) | QR factorization of matrices |
| 4. Curtiss & Hirschfelder (1952) | stiffness of ODEs; BD formulas |
| 5. de Boor (1972) | calculations with B-splines |
| 6. Courant (1943) | finite element methods for PDE |
| 7. Golub & Kahan (1965) | the singular value decomposition |
| 8. Brandt (1977) | multigrid algorithms |
| 9. Hestenes & Stiefel (1952) | the conjugate gradient iteration |
| 10. Fletcher & Powell (1963) | optimization via quasi-Newton updates |
| 11. Wanner, Hairer & Norsett (1978) | order stars and applications to ODE |
| 12. Karmarkar (1984) | interior pt. methods for linear prog. |
| 13. Greengard & Rokhlin (1987) | multipole methods for particles |

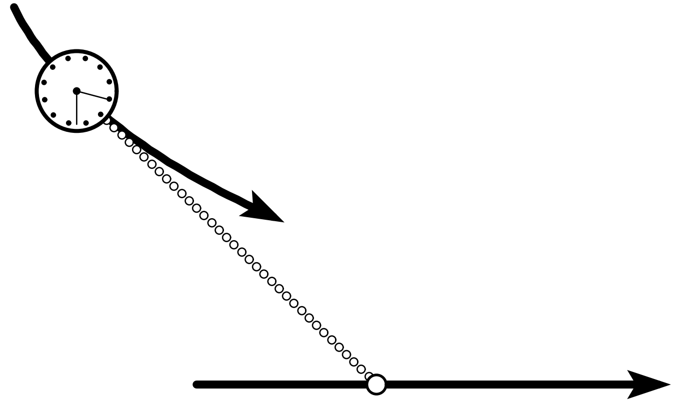
# From where come the Mathematical Models ??



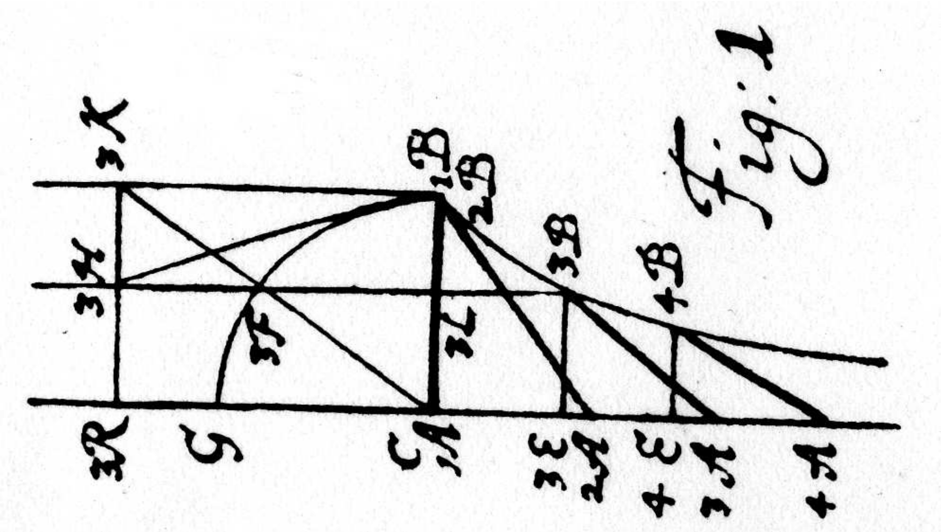

want to go below the picture...

## 2. Inverse Numerical Analysis. Some examples ...

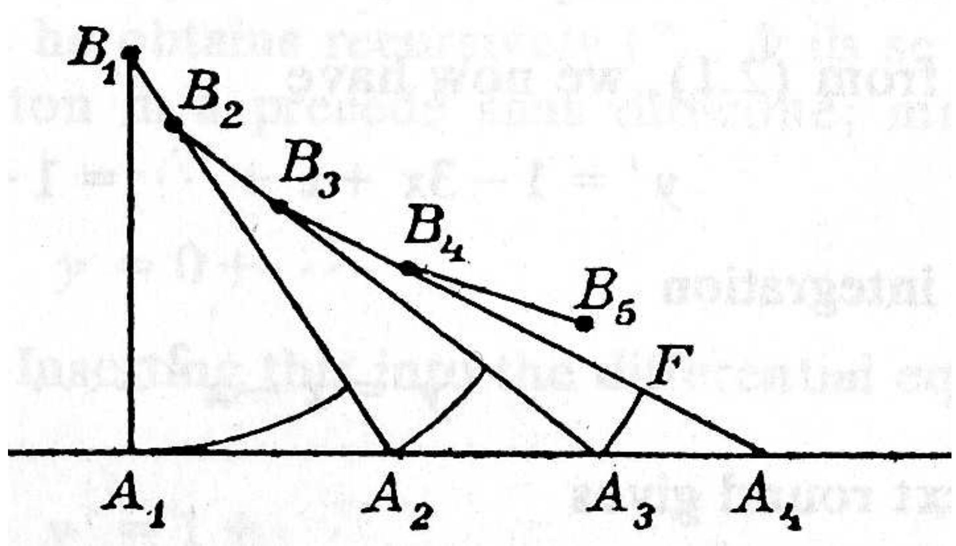
# Leibniz 1674, 1693: the Tractrix.



“horologio portabili suae thecae argentae”



Drawing Leibniz 1693

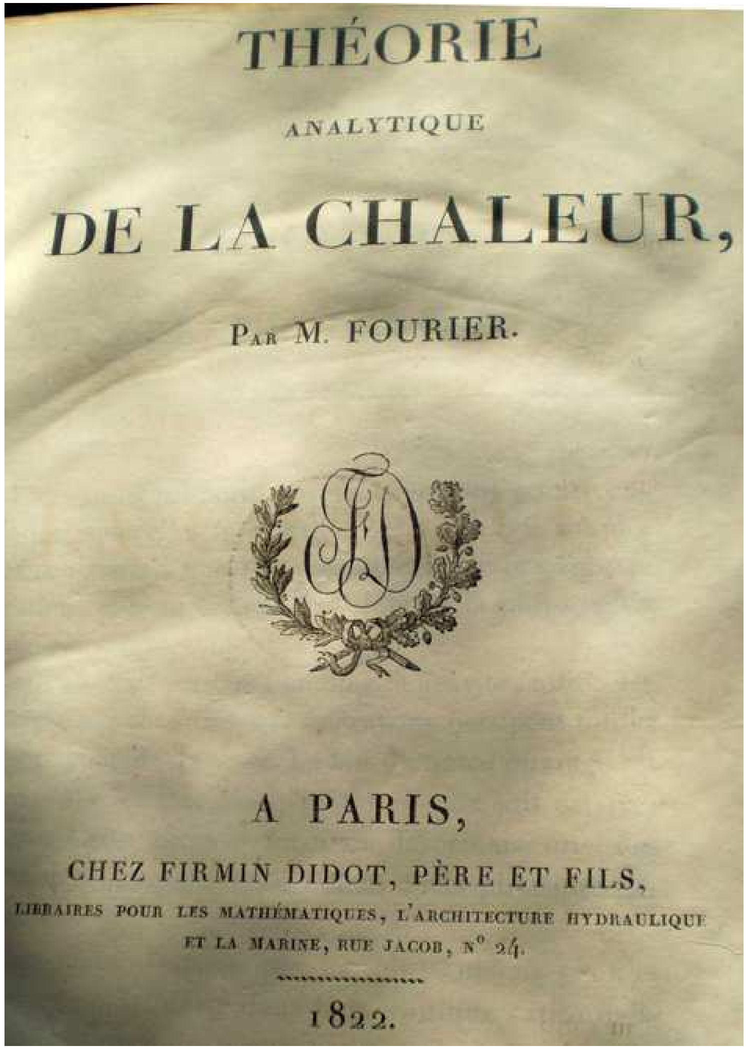



Drawing Kowalewski 1908

Physical intuition and inverse backward Euler method tell us :

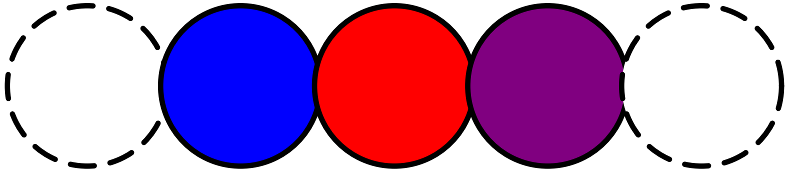
$$\text{curve is solution of} \quad \frac{dy}{dx} = -\frac{y}{\sqrt{a^2 - y^2}}\,.$$

# Fourier's Heat Equation (1807, 1822)


THÉORIE
ANALYTIQUE
DE LA CHALEUR,
PAR M. FOURIER.
A PARIS,
CHEZ FIRMIN DIDOT, PÈRE ET FILS,
LIBRAIRES POUR LES MATHÉMATIQUES, L'ARCHITECTURE HYDRAULIQUE
ET LA MARINE, RUE JACOB, N° 24.
1822.

59.

En désignant par $v$ et $v'$ les températures des deux molécules égales $m$ et $n$; par $p$, leur distance extrêmement petite, et par $dt$, la durée infiniment petite de l'instant, la quantité de chaleur que $m$ reçoit de $n$, pendant cet instant, sera exprimée par $(v'-v)\,\varphi(p).dt$.

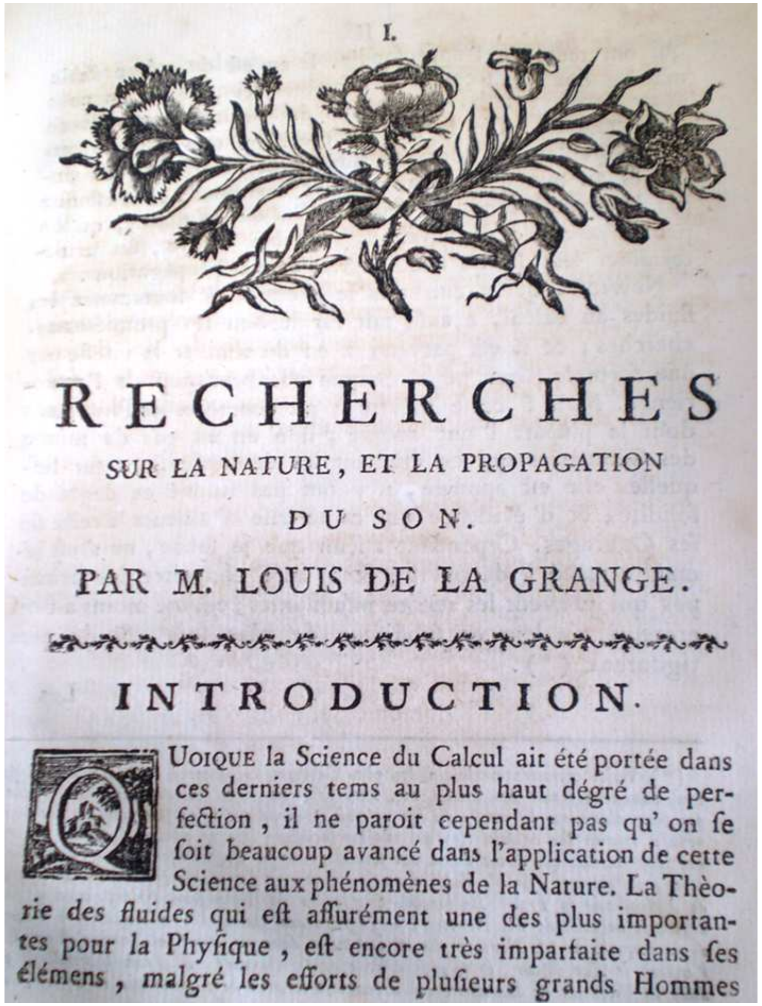



$$\frac{dv_i}{dt} = K \cdot \left( (v_{i+1} - v_i) - (v_i - v_{i-1}) \right)$$

$$\frac{dv}{dt} = \frac{K}{CD} \frac{d^2v}{dx^2}.$$

Heat equation from inverse method of lines.

# Lagrange’s theory of Sound (Miscell. Taurinensia 1, 1759)



I.

RECHERCHES

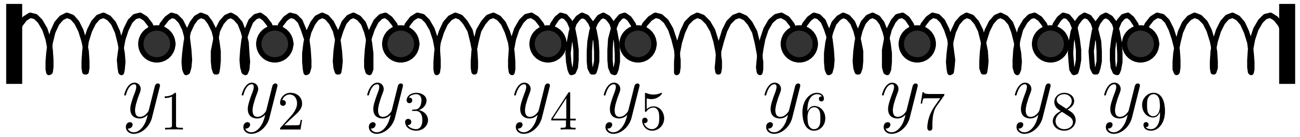
SUR LA NATURE, ET LA PROPAGATION

DU SON.

PAR M. LOUIS DE LA GRANGE.

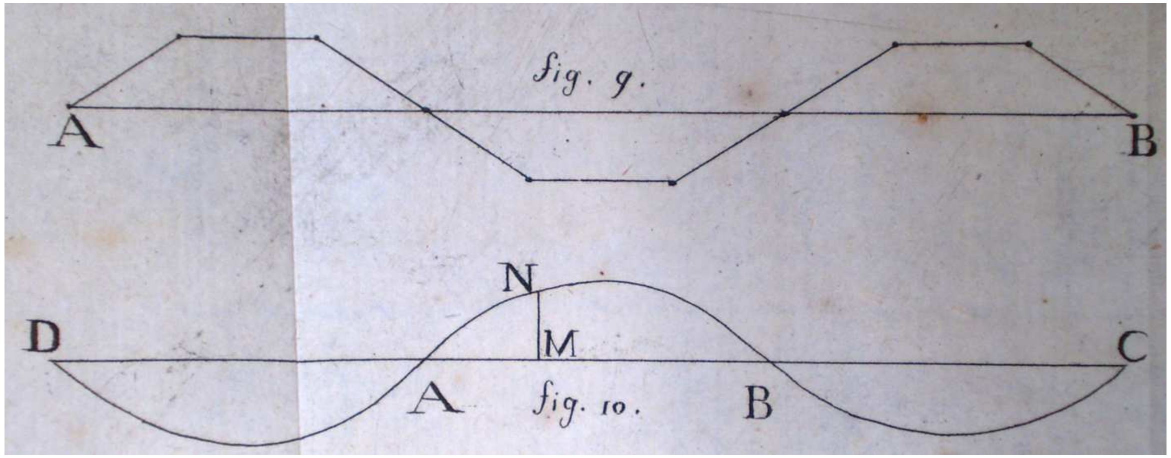
INTRODUCTION.

Quoique la Science du Calcul ait été portée dans ces derniers tems au plus haut dégré de perfection, il ne paroit cependant pas qu' on se soit beaucoup avancé dans l'application de cette Science aux phénomènes de la Nature. La Théorie des fluides qui est assurément une des plus importantes pour la Physique, est encore très imparfaite dans ses élémens, malgré les efforts de plusieurs grands Hommes




$$\frac{d^2 y^m}{dt^2} = \frac{2\,A\,h}{T^2} \times \frac{y^{m+1} - 2\,y^m + y^{m-1}}{r^2}$$

$$\left(\frac{d^2 y}{dt^2}\right) = c\left(\frac{d^2 y}{dx^2}\right)$$

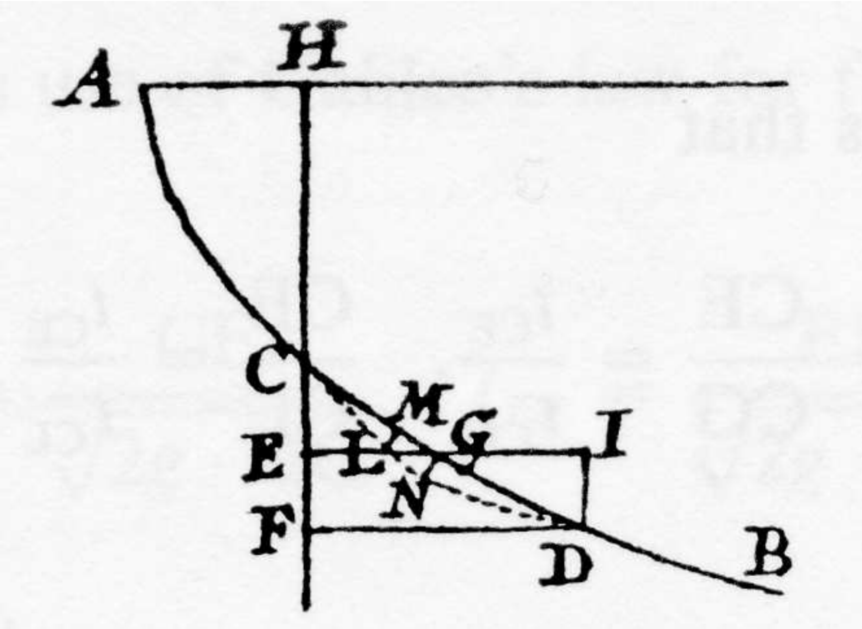



PDE for sound from inverse method of lines.

# Variational Calculus.

## Challenge of Joh. Bern. (1696)

$$\int \frac{\sqrt{dx^2 + dy^2}}{\sqrt{y}} = \text{min!}$$

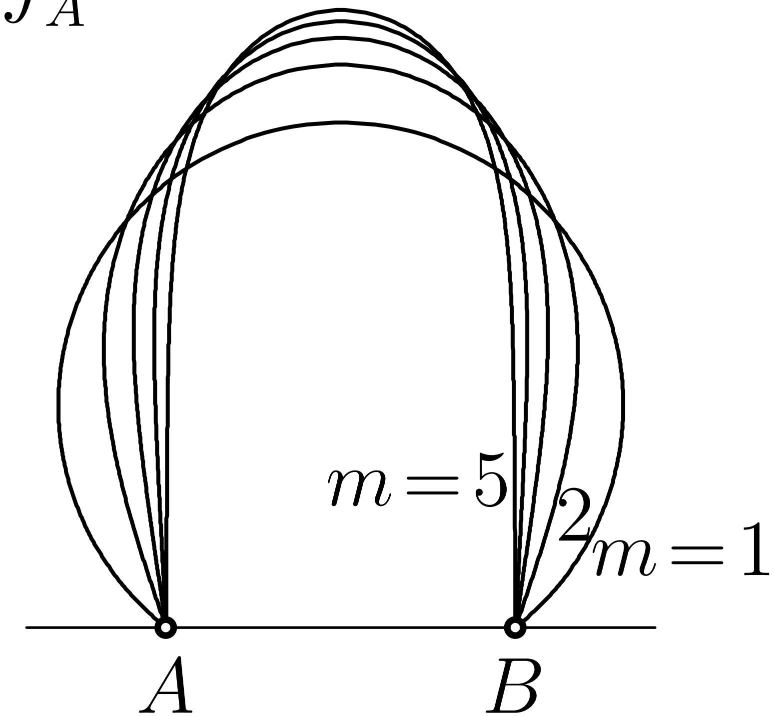



Jakob could solve.

## Challenge of Jak. Bern. (1697)

$$\int_A^B y^m \, dx = \text{max!}$$

$$\int_A^B ds = L \quad \text{(given)}$$

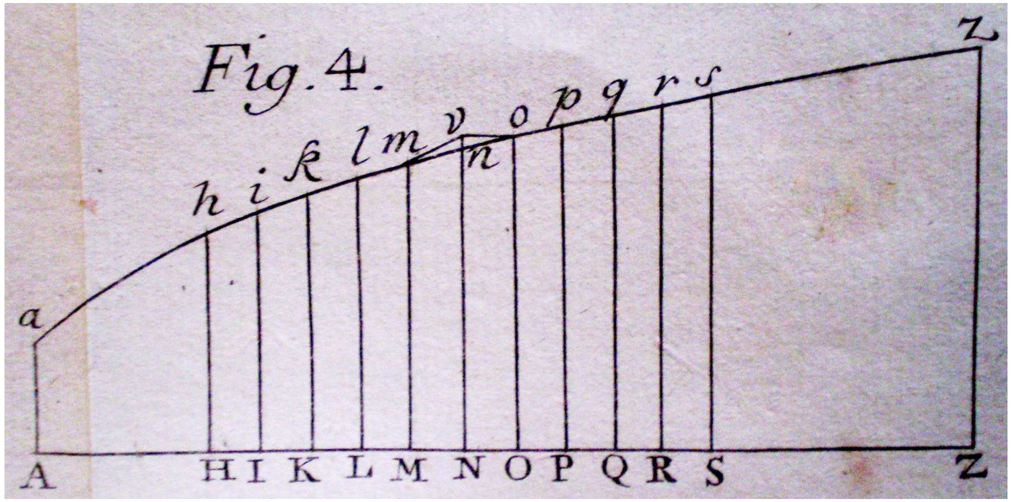



Joh. could not solve.

**Euler** (E65, 1744):

$$J=\int_a^b Z\,dx = \text{min! vel max!} \quad \text{where } Z=f(x,y,p),\ \ p=\frac{dy}{dx}\,.$$

**Euler's Solution.**




1. Approx. curve by polygon (today: "FE-Method")

2. Approx. Integral by -sum ("Riemann")

$$Z\,dx + Z'\,dx + Z''\,dx + Z'''\,dx + \&c.$$

and diff. w.r. to $\nu$;

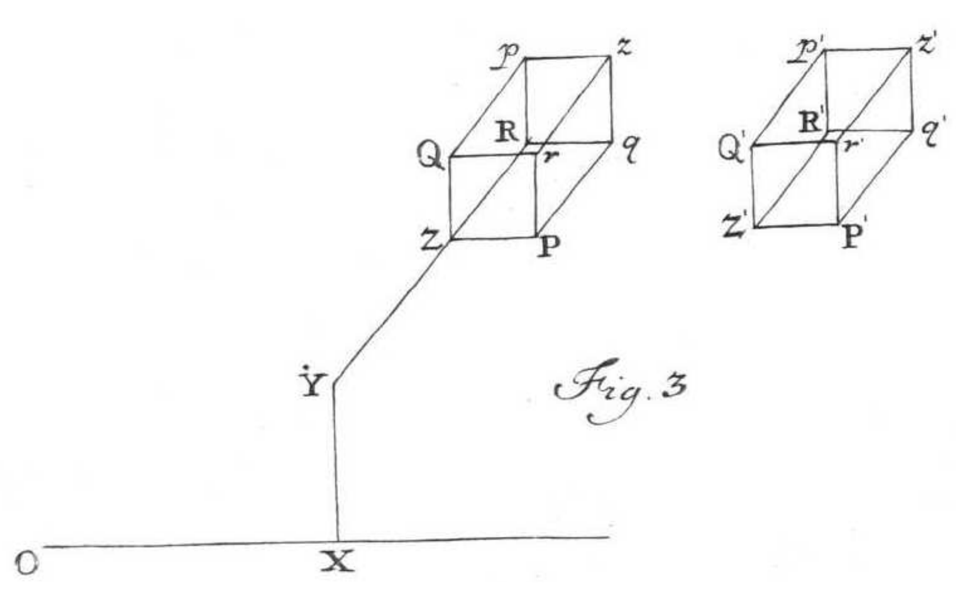
3. Set derivative $(P + N'dx - P')$ to zero;

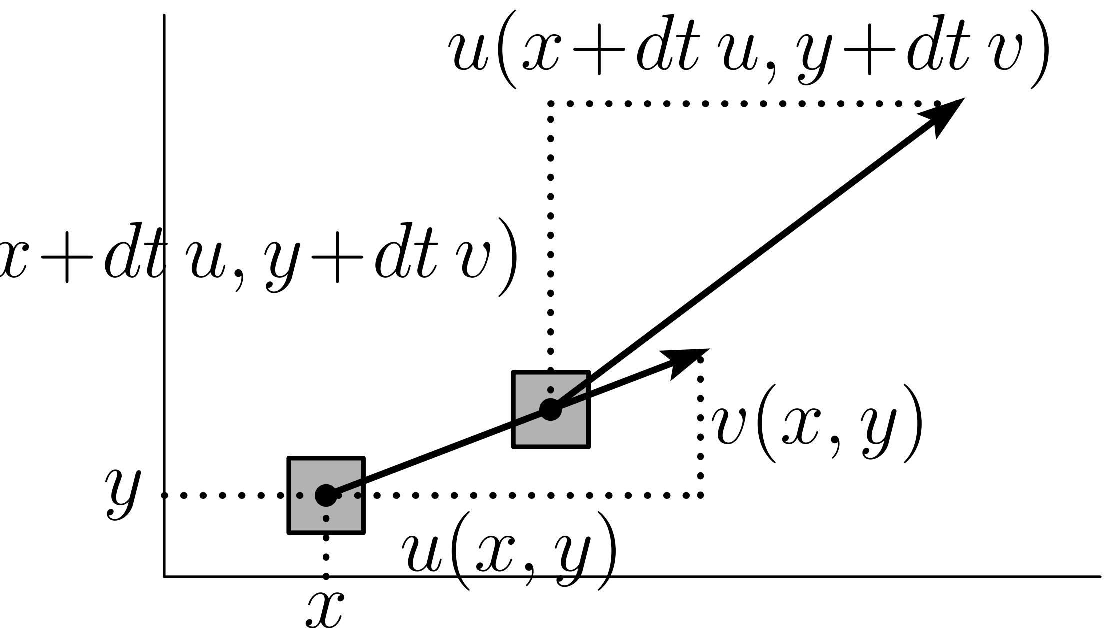
4. inverse Euler method $\Rightarrow$

$$N - \frac{dP}{dx} = 0$$

$\left(N=\frac{\partial f}{\partial y},\right.$ $\left.P=\frac{\partial f}{\partial p}\right).$

# **Hydrodynamics** (Euler E226, E227, 1753)




$$\begin{pmatrix} du \\ dv \end{pmatrix} = \begin{pmatrix} u(x+u\,dt, y+v\,dt)-u(x,y) \\ v(x+u\,dt, y+v\,dt)-v(x,y) \end{pmatrix}$$

$$= \begin{pmatrix} u\cdot\frac{\partial u}{\partial x}+v\cdot\frac{\partial u}{\partial y} \\ u\cdot\frac{\partial v}{\partial x}+v\cdot\frac{\partial v}{\partial y} \end{pmatrix} dt$$

"Nous n'avons donc qu'à égaler ces forces accélératrices avec les accélérations actuelles."

finite diff. $\Rightarrow$ Euler's eq. for FD.

**Theory of Numerical Methods**

More respectability

**“Brute” construction of Numerical Methods**

Numerical methods

**Mathematical Problems to Solve**

INVERSE Numerical methods

**Basic Principles, Physical Intuition, Experiments**

Although this may seem a paradox, all exact science is dominated by the idea of approximation.

(Bertrand Russell; found by Jan Lacki)

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(Bertrand Russell; found by Jan Lacki)

# 3. Mechanics and the Law of Gravitation.

(Gravity ...) was one of the first great laws to be discovered and it has an interesting history. You may say, 'Yes, but then it is old hat, I would like to hear something about a more modern science'. More recent perhaps, but not more modern. (...) I do not feel at all bad about telling you about the Law of Gravitation, because in describing its history and methods, the character of its discovery, its quality, I am being completely modern.

(R. Feynman 1965)

"Astronomy is older than physics. In fact, it got physics started by showing the beautiful simplicity of the motion of the stars and planets, the understanding of which was the *beginning* of physics."

(R. Feynman 1963)

Babylonian Priests $\Rightarrow$ Greek Astronomers (Ptolemy)
$\Rightarrow$ Arabian Astronomers $\Rightarrow$ Regiomontanus
$\Rightarrow$ Copernicus $\Rightarrow$ Tycho Brahe $\Rightarrow$ Kepler
$\Rightarrow$ Descartes $\Rightarrow$ Newton $\Rightarrow$ Euler $\Rightarrow$ $\ldots$

“Astronomy is older than physics. In fact, it got physics started by showing the beautiful simplicity of the motion of the stars and planets, the understanding of which was the *beginning* of physics.”

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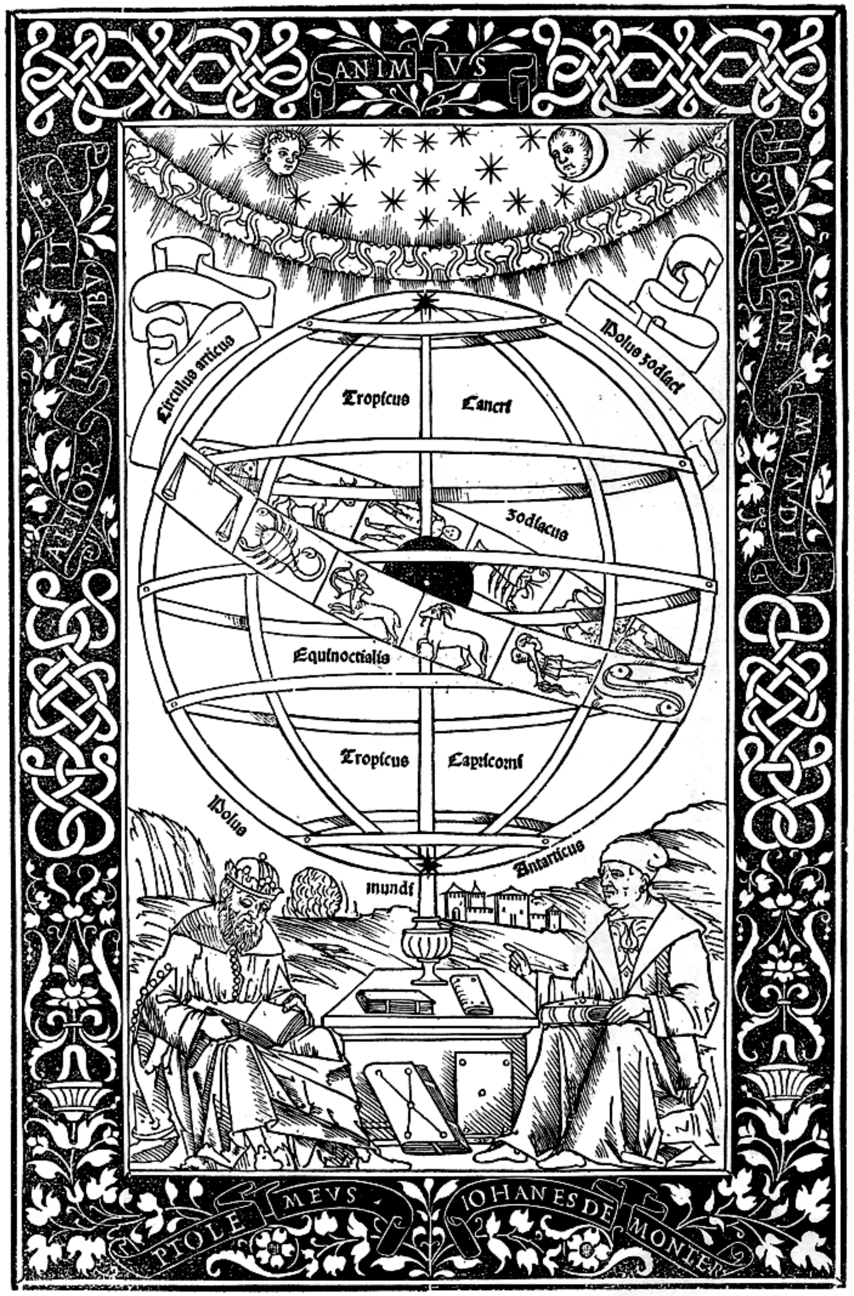
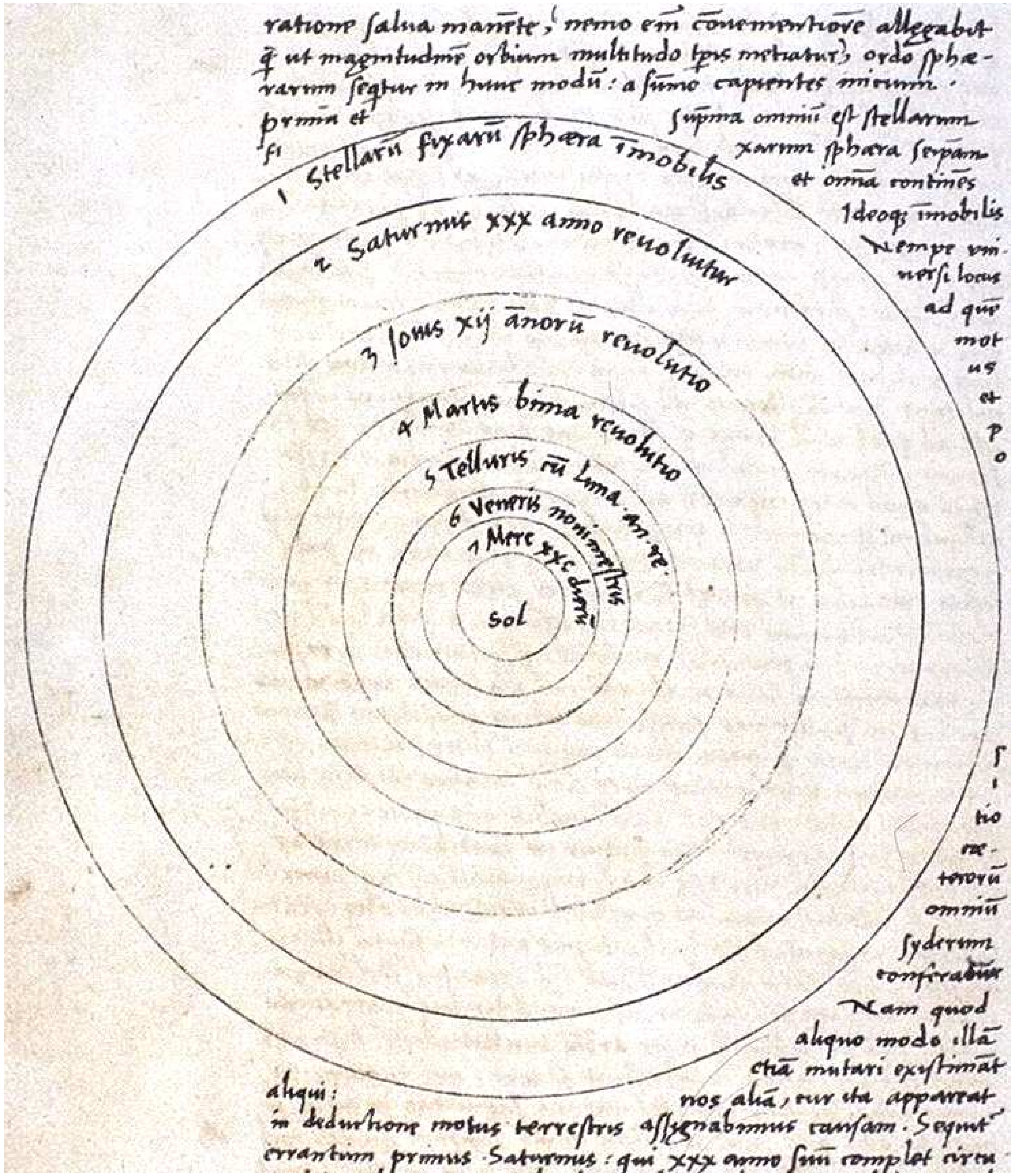
$\Rightarrow$ Arabian Astronomers $\Rightarrow$ Regiomontanus
$\Rightarrow$ Copernicus $\Rightarrow$ Tycho Brahe $\Rightarrow$ Kepler **1609**
$\Rightarrow$ Descartes $\Rightarrow$ Newton $\Rightarrow$ Euler $\Rightarrow$ . . .

**400 years of modern science**

# The Birth of Science




(Ptolemy 150 A.D., publ. 1496)




(Copernicus, publ. 1543)

# Tycho Brahe (1546 – 1601):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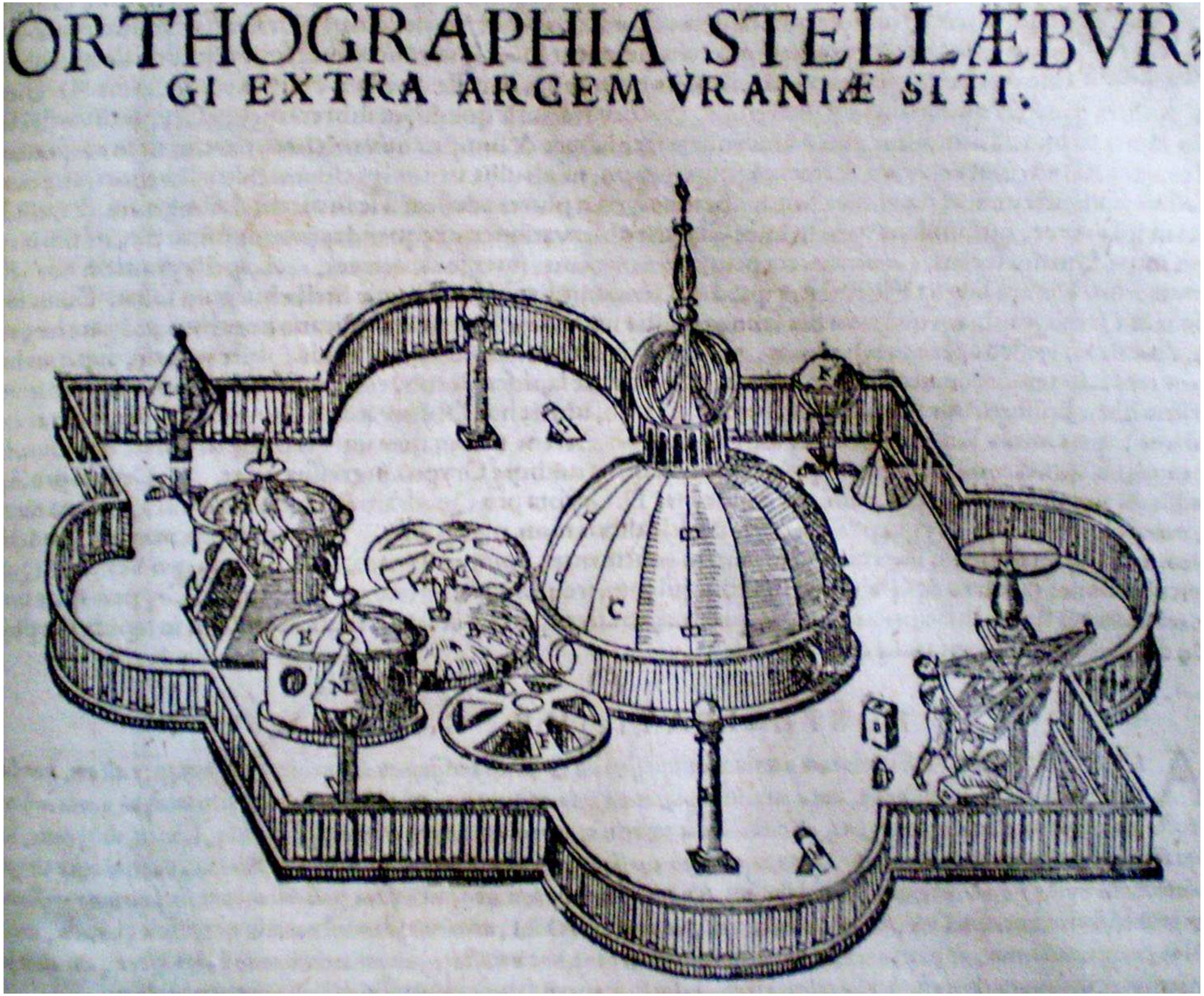
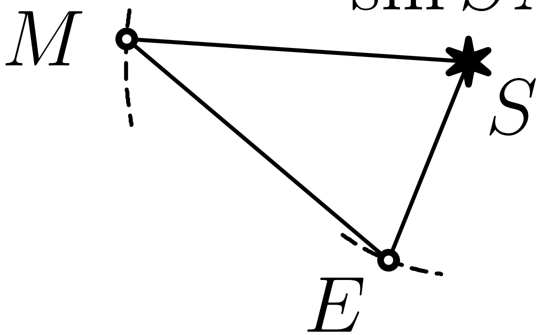



measured thousands of angles with unequalled precision;

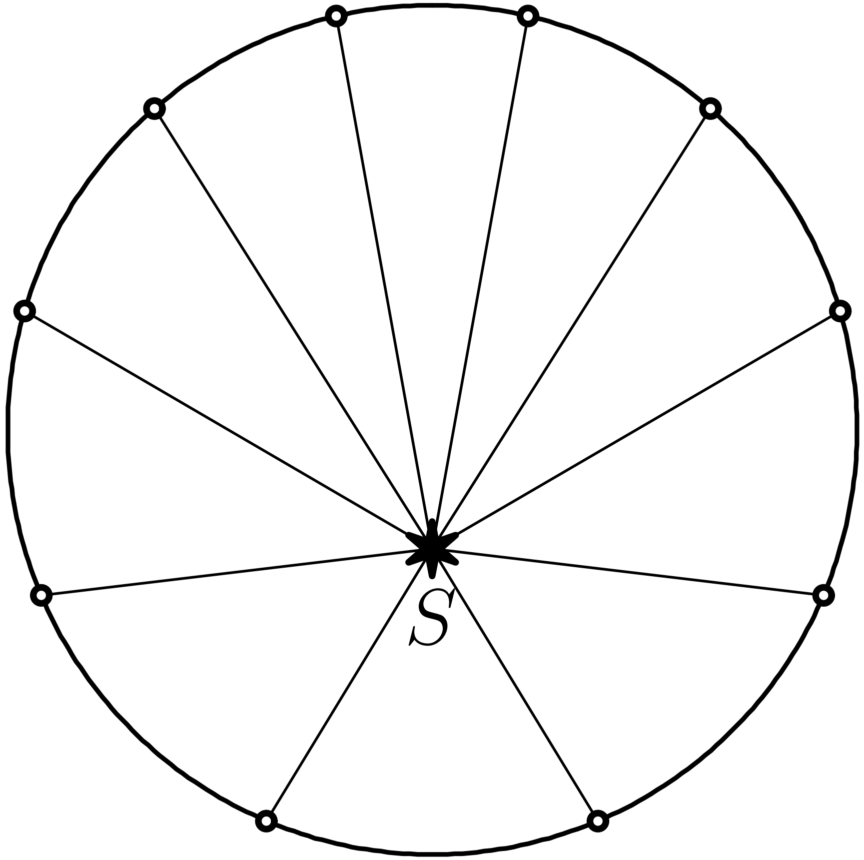
**Tycho Brahe** (1546 – 1601); back to Ptolemy's system.

computed thousands of distances

$$SM = SE \cdot \frac{\sin SEM}{\sin SME}\;;$$




$\Rightarrow$ orbits are circles !!

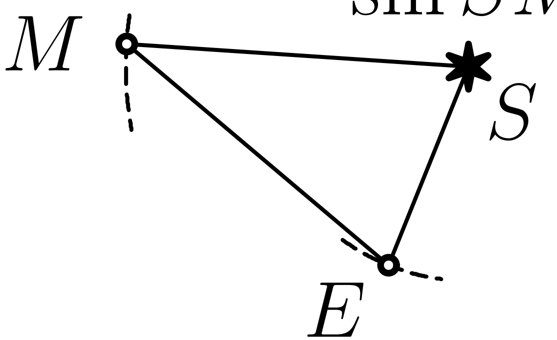



$S$: Sun.

# **Tycho Brahe** (1546 – 1601); back to Ptolemy's system.

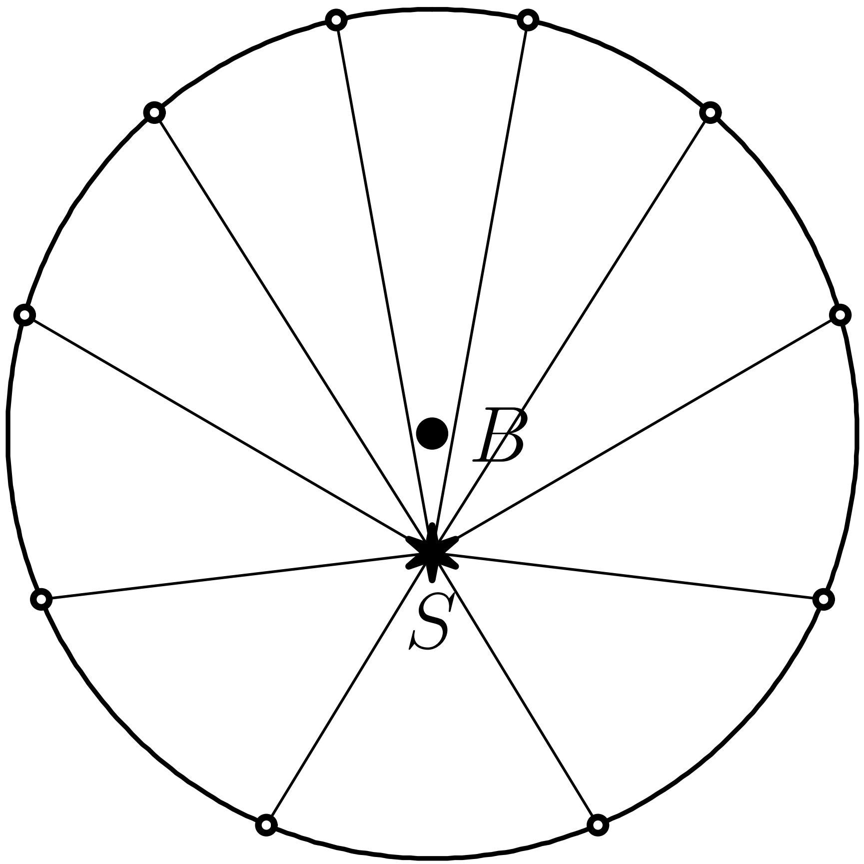
computed thousands of distances

$$SM = SE \cdot \frac{\sin SEM}{\sin SME} \; ;$$




$\Rightarrow$ orbits are excentric circles !!

(Inaequalitatis primae)




$S$: Sun,

$B$: "Mean" Sun.

# **Tycho Brahe** (1546 – 1601); back to Ptolemy's system.

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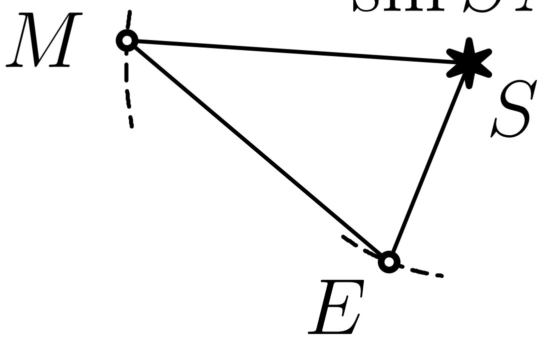
$$SM = SE \cdot \frac{\sin SEM}{\sin SME}\ ;$$

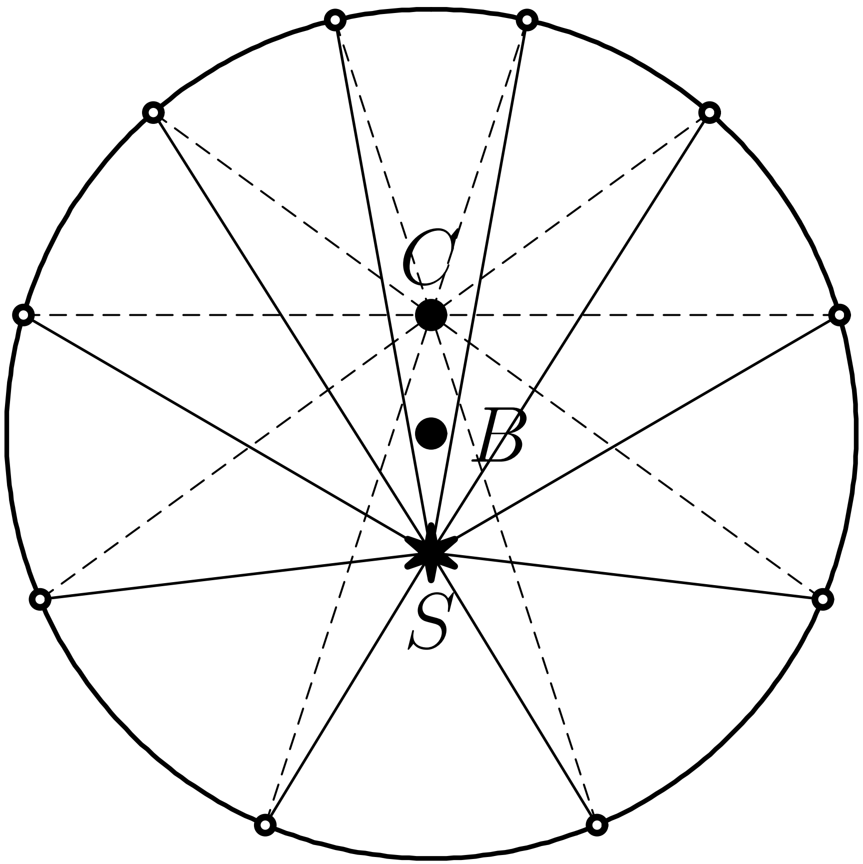



⇒ orbits are excentric circles !!

(Inaequalitatis primae)

"punctum aequans" governs speed

(Inaequalitatis secundae)




$S$: Sun,

$B$: "Mean" Sun,

$C$: punctum aequans.

**Tycho Brahe** (1546 – 1601); back to Ptolemy's system.

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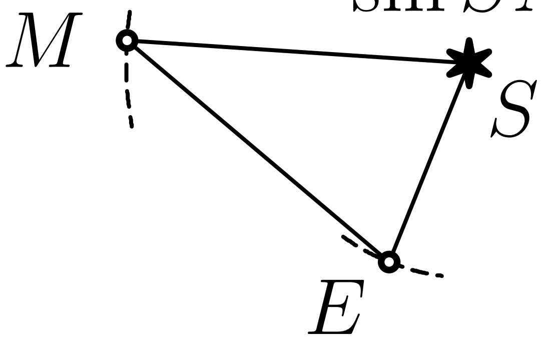
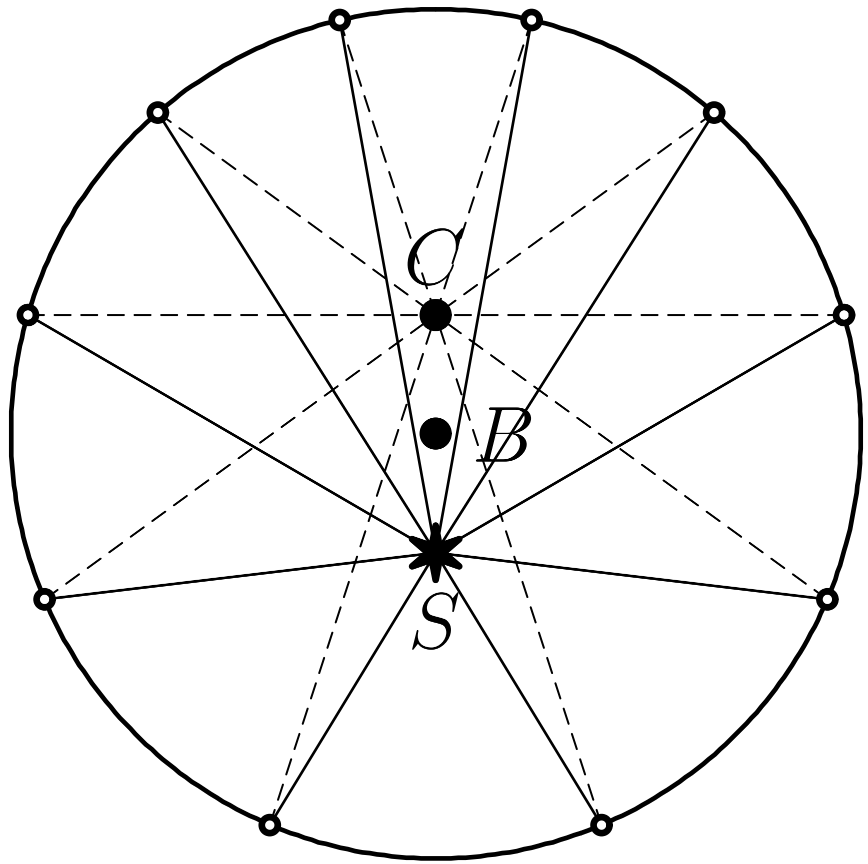
$$SM = SE \cdot \frac{\sin SEM}{\sin SME} \, ;$$



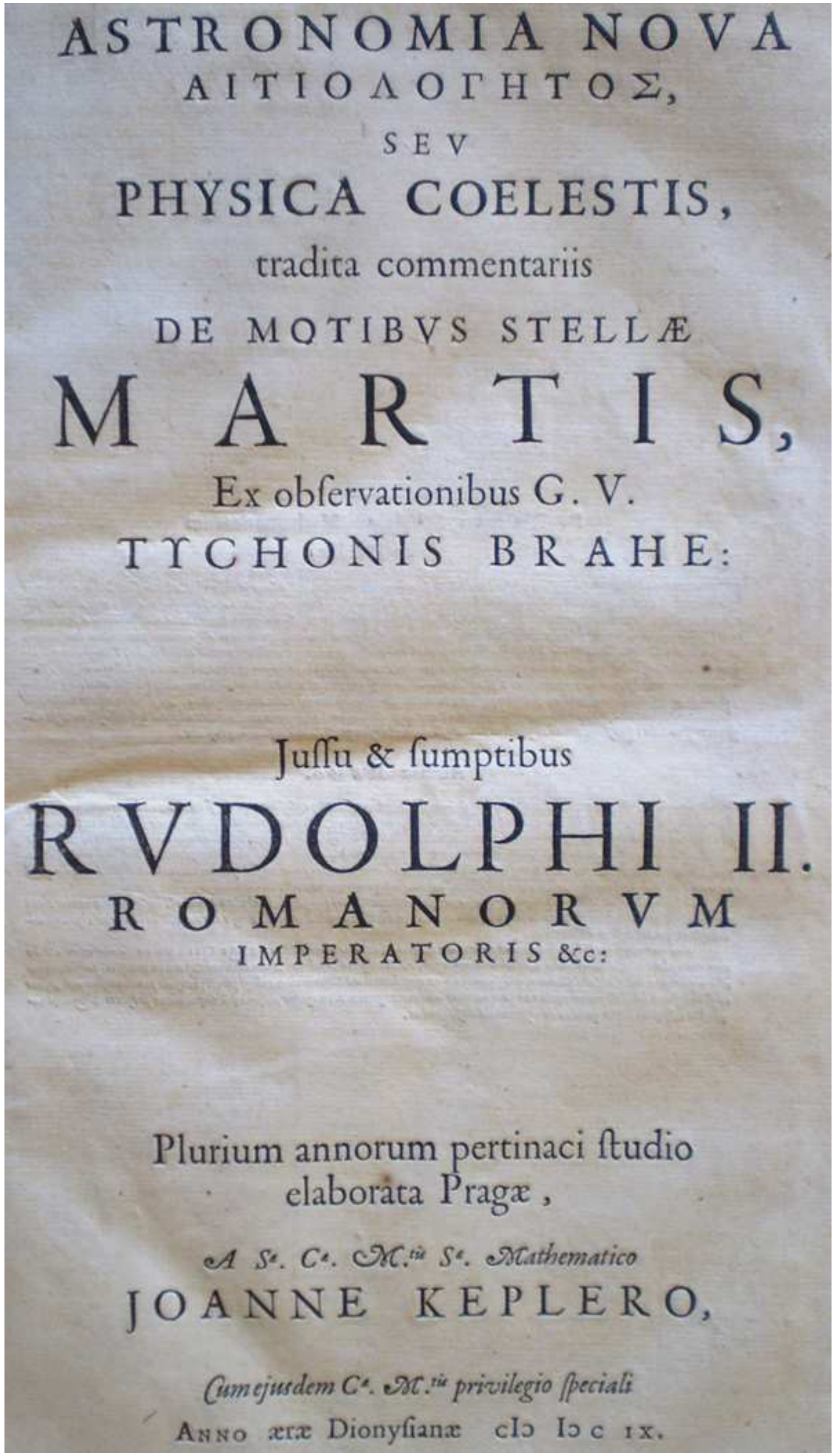

$S$: Sun,
$B$: "Mean" Sun,
$C$: punctum aequans.

⇒ orbits are excentric circles !!
(Inaequalitatis primae)

"punctum aequans" governs speed
(Inaequalitatis secundae)

Worked fine for all planets **except Mars !!**

# Johannes Kepler (1609): Great Triumph:



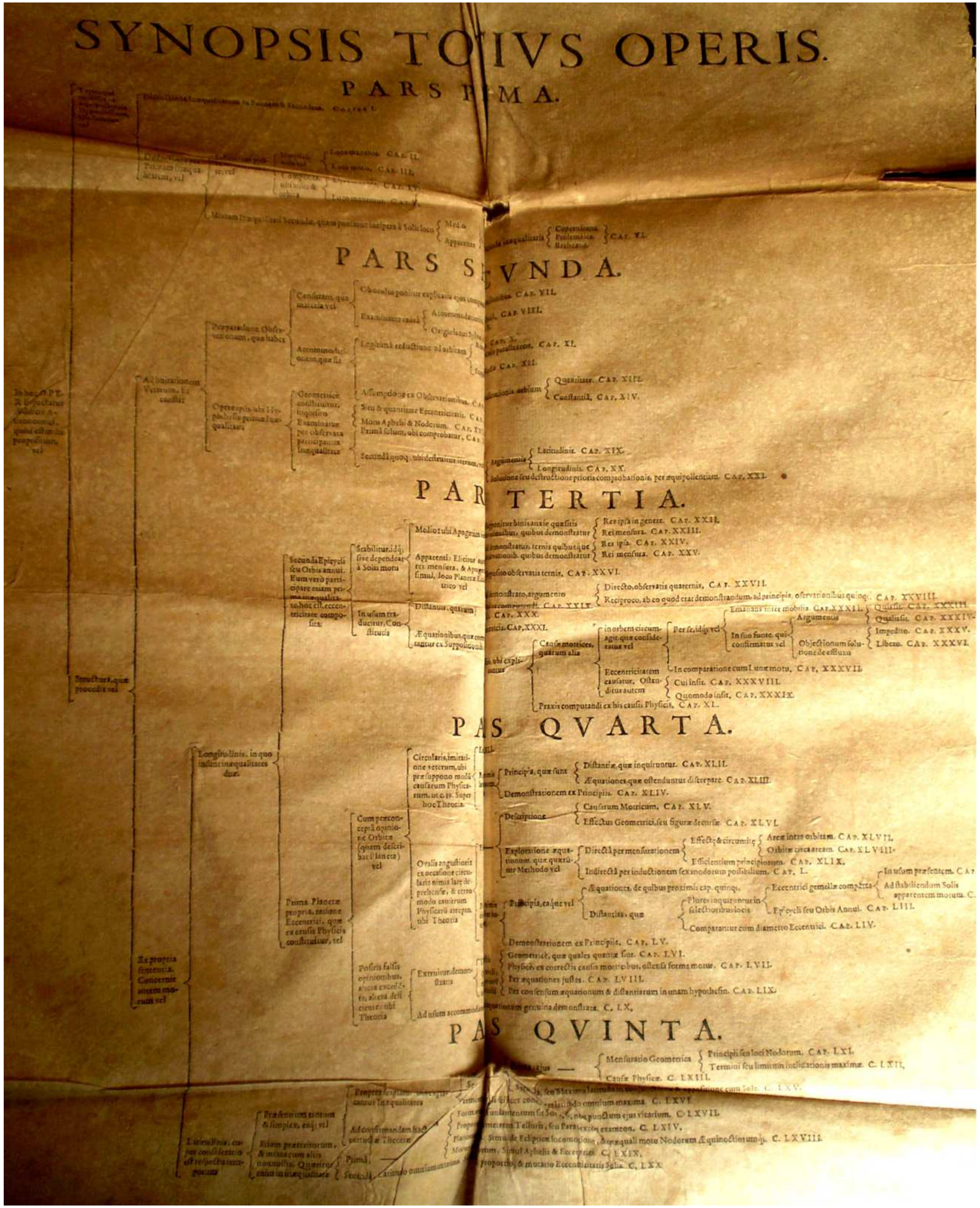
ASTRONOMIA NOVA
ΑΙΤΙΟΛΟΓΗΤΟΣ,
SEV
PHYSICA COELESTIS,
tradita commentariis
DE MOTIBVS STELLÆ
MARTIS,
Ex obſervationibus G. V.
TYCHONIS BRAHE:

Juſſu & ſumptibus
RVDOLPHI II.
ROMANORVM
IMPERATORIS &c:

Plurium annorum pertinaci ſtudio
elaborata Pragæ,

A S^e. C^a. M.^tis S^e. Mathematico
JOANNE KEPLERO,

Cum ejusdem C^a. M.^tis privilegio ſpeciali
ANNO æræ Dionyſianæ CIƆ IƆ C IX.



The book starts with

Very hard is today the condition for writing mathematical books, especially astronomical ones. If you don't preserve all the propositions, arrangements, demonstrations, conclusions, the book would not be mathematical; on the other side, if you preserve them, the book is very boring to read ...

# INTRODVCTIO
## IN HOC OPVS.

DVRISSIMA EST HODIE CONDITIO SCRIBENDI LIBROS Mathematicos, præcipue Astronomicos. Nisi enim servaveris genuinam subtilitatem propositionum, instructionum, demonstrationum, conclusionum; liber non erit Mathematicus: sin autem servaveris; lectio efficitur morosissima, præsertim in Latina lingua, quæ caret articulis, & illa gratia quam habet græca, cum per signa literaria loquitur. Adeoq; hodie perquam pauci sunt lectores idonei: cæteri in commune respuunt. Quotusquisq; Mathematicorum est, qui tolerat laborem perlegendi Appollonii Pergæi Conica? Est tamen illa materia ex eo rerum genere, quod longe facilius exprimitur figuris & lineis quam Astronomica.

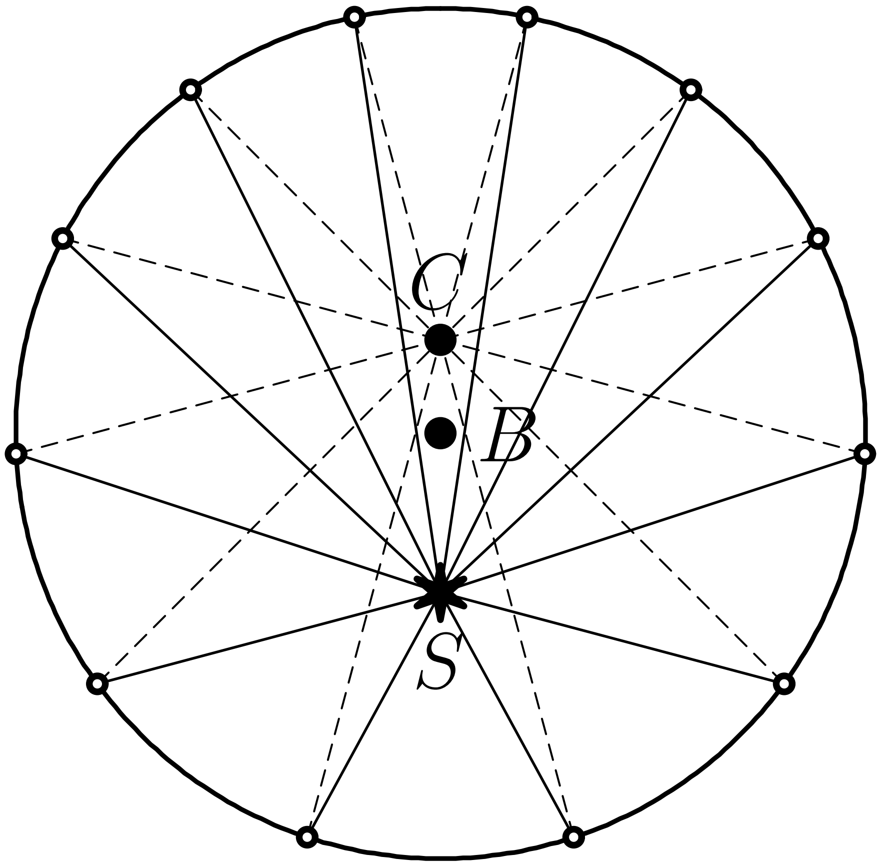
## Kepler's Pars Secunda : "Ad imitationem veterum"

(the "Ancients" are Ptolemy, Copernicus and Tycho Brahe, 25 years older than he).

Let the "punctum aequans" $C$ have an arbitrary position;

determine $S$, $B$ and $C$ after very "disgusting" calculations from 4 observations (solve numer. 4-dim. nonl. system)

$\Rightarrow$ Hypothesis vicaria !!




If you find these calculations disgusting (pertaesum), then have pity for me (jure mei te miserat), I did them at least 70 times losing a lot of time (ad minimum septuagies ivi cum plurima temporis jactura)...

**Pars Tertia :** “Ex propria sententia” (his own opinion)

Voilà les fondements de la physique moderne, céleste et terrestre.

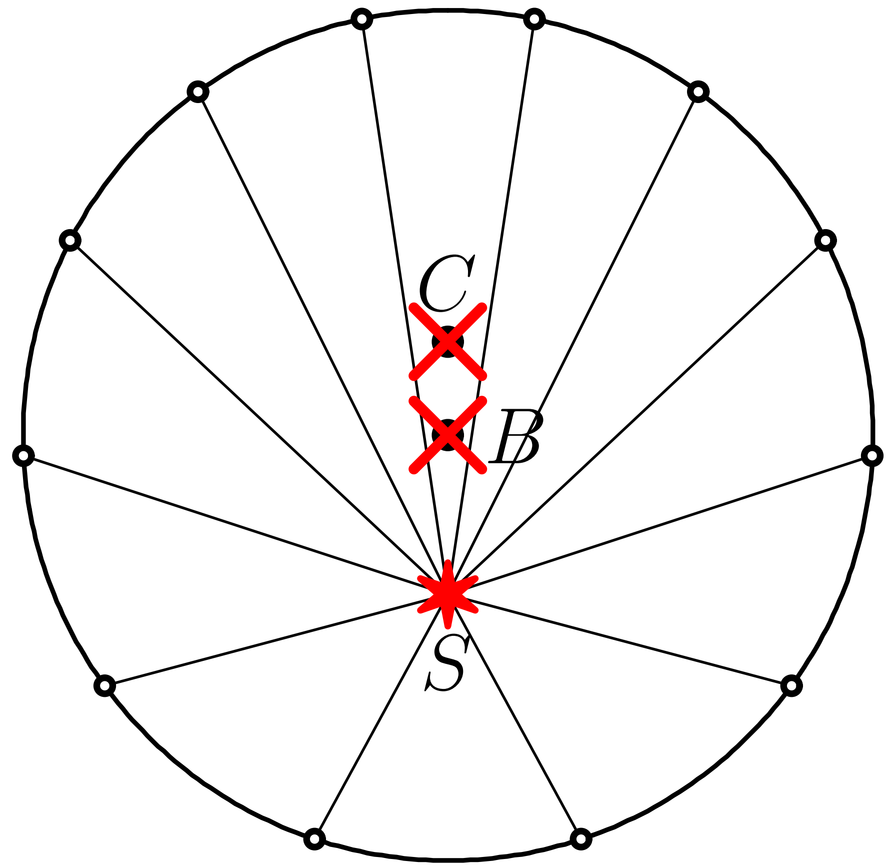
(Delambre 1821; see M. Caspar, *J. Kepler, Werke 3*, p. 256)

Kepler, as convinced Copernician,
puts **Sun** in the center;
get rid of $B$ and $C$,
which have no physical meaning:

ΑΙΤΙΟΛΟΓΗΤΟΣ

PHYSICA COELESTIS,

But how fast move the Planets ??



## **Pars Tertia :** “Ex propria sententia” (his own opinion)

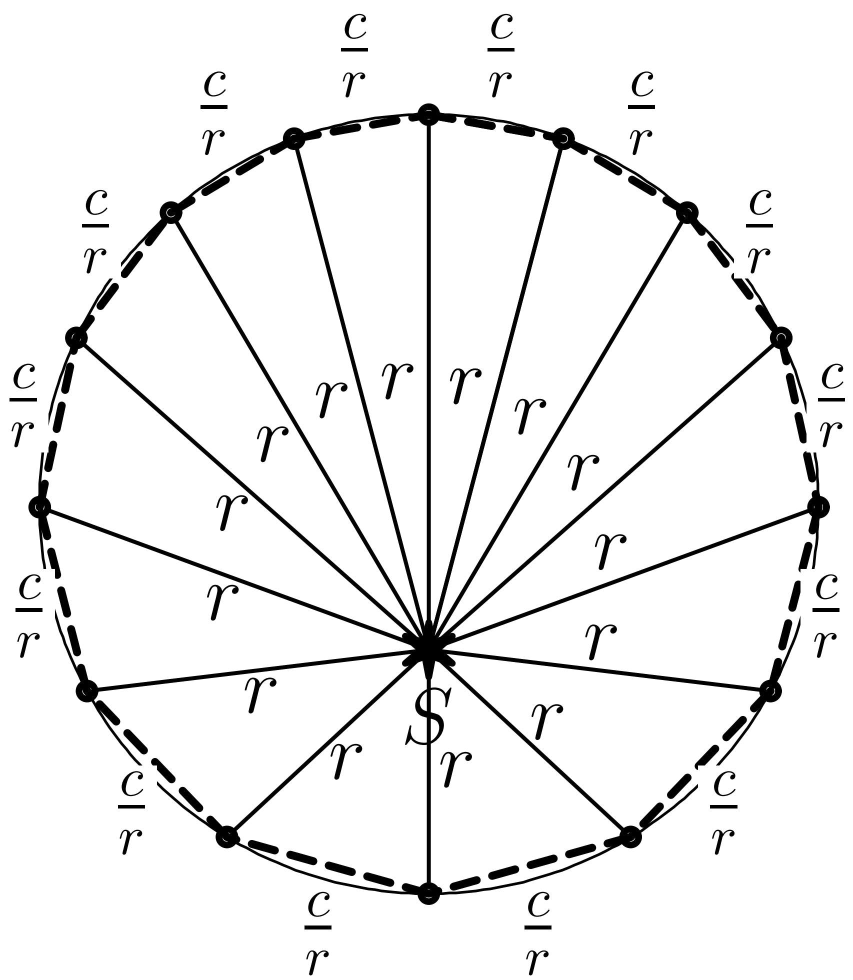
Voilà les fondements de la physique moderne, céleste et terrestre.

(Delambre 1821; see M. Caspar, *J. Kepler, Werke 3*, p. 256)

Long discussions,
(Chap. 32–39)
attractive forces, magnetism,
the planets have a “Soul” ;
the planets “wish” to move;
the planets “look” at the Sun and
see diameter inv. prop. to $r$

$\Rightarrow$ Speed inversely prop. to $r$ !!

(New Inaequalitatis secundae).



# Chap. 40: End of Pars Tertia : Simplified model:

CAPVT XL.

Methodus imperfecta æquationes ex Phyſica hypotheſi computandi, quæ tamen ſufficit theoriæ SOLIS vel TERRÆ.

$\Rightarrow$ Replace hypothenuse by leg

diſtantias omnes ineſſe. Nam memineram, ſic olim & ARCHIMEDEM, cum circumferentiæ proportionem ad diametrum quæreret, circulum in infinita triangula diſſecuiſſe. nam hæc vis occulta eſt ejus demon-

$\Rightarrow$ all triangles have same area !!

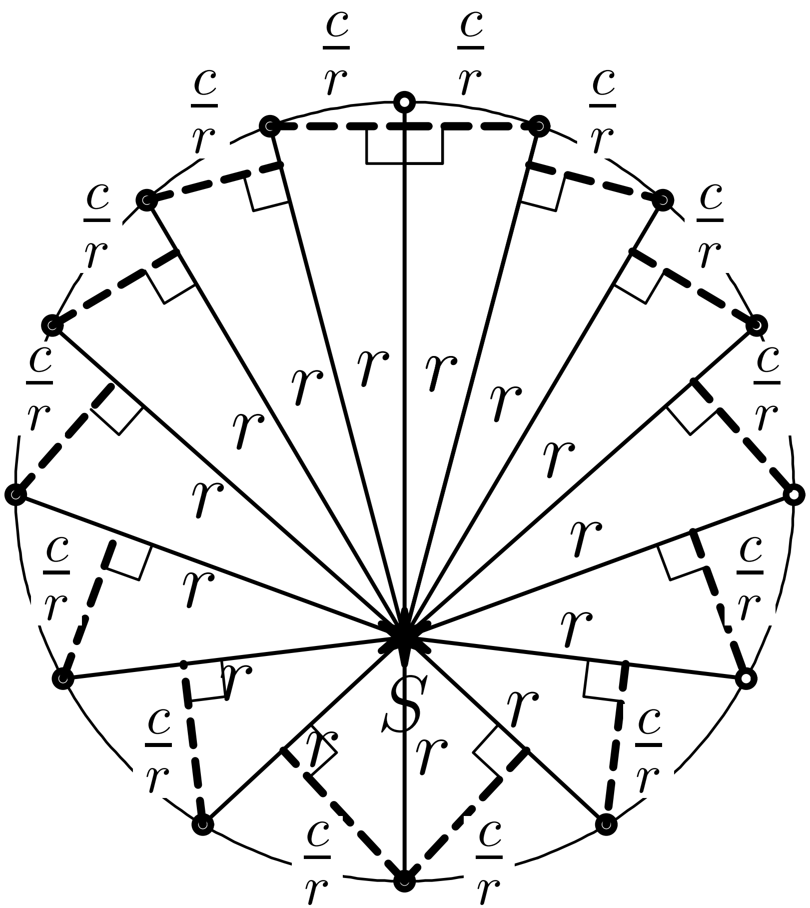
(Correct Inaequalitatis secundae)

What Kepler thought was right

— is wrong;

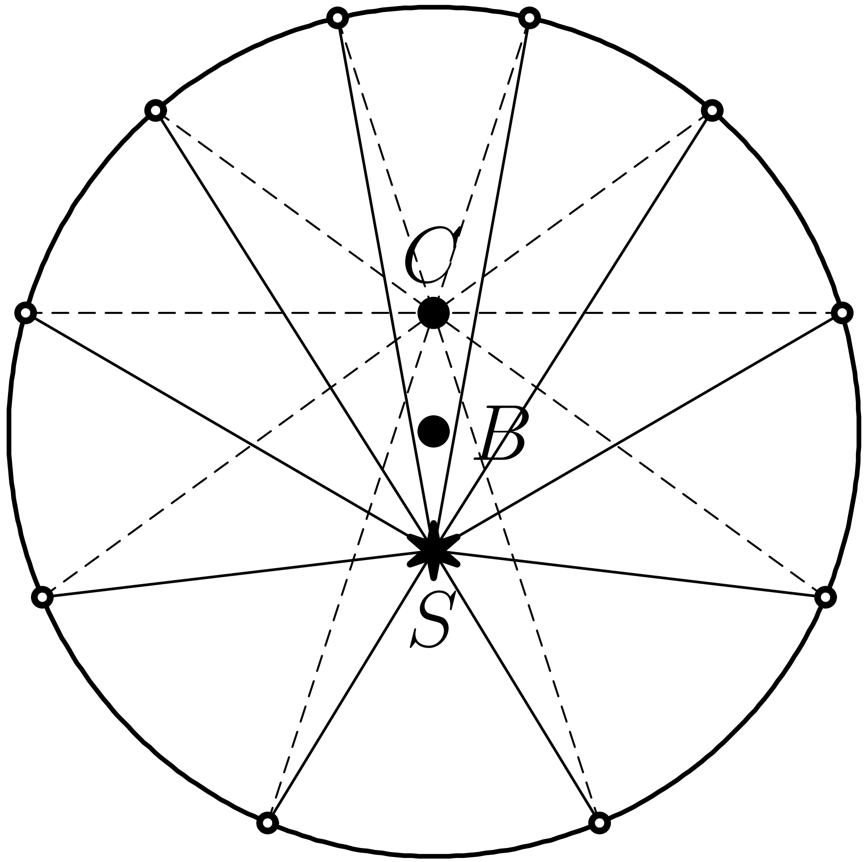
What Kepler thought was wrong

— is right !!

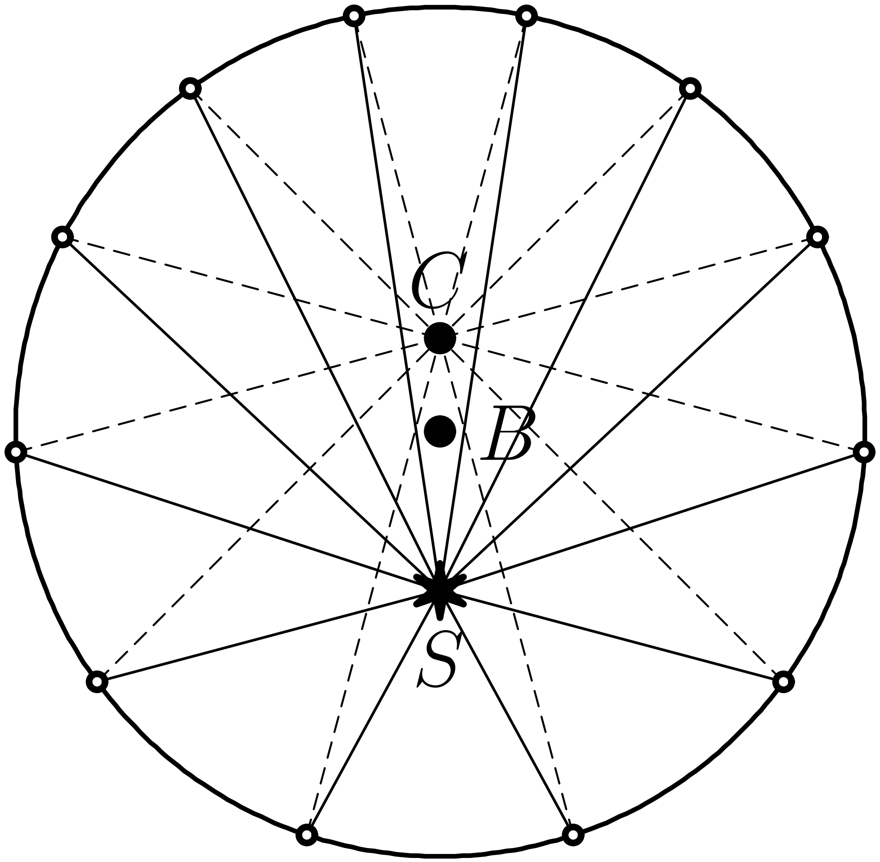


# Recapitulation of Discovery of Kepler's Second Law :

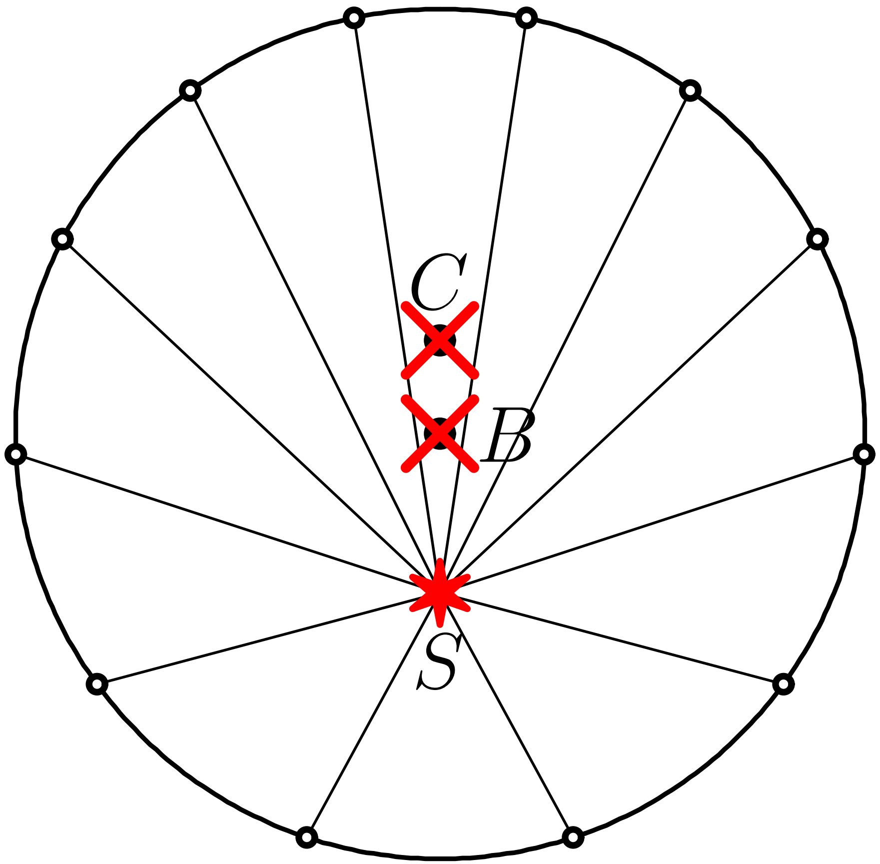
Ptolemy, Copernicus, Brahe:



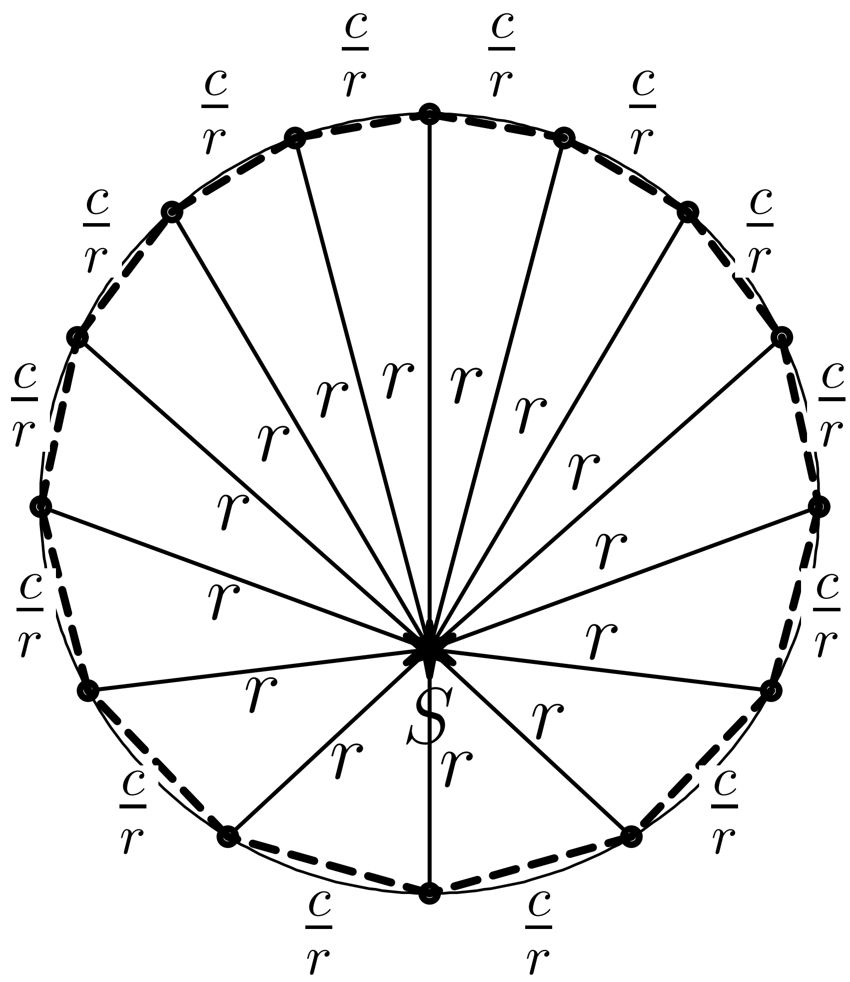

Kepler, Hypothesis vicaria:



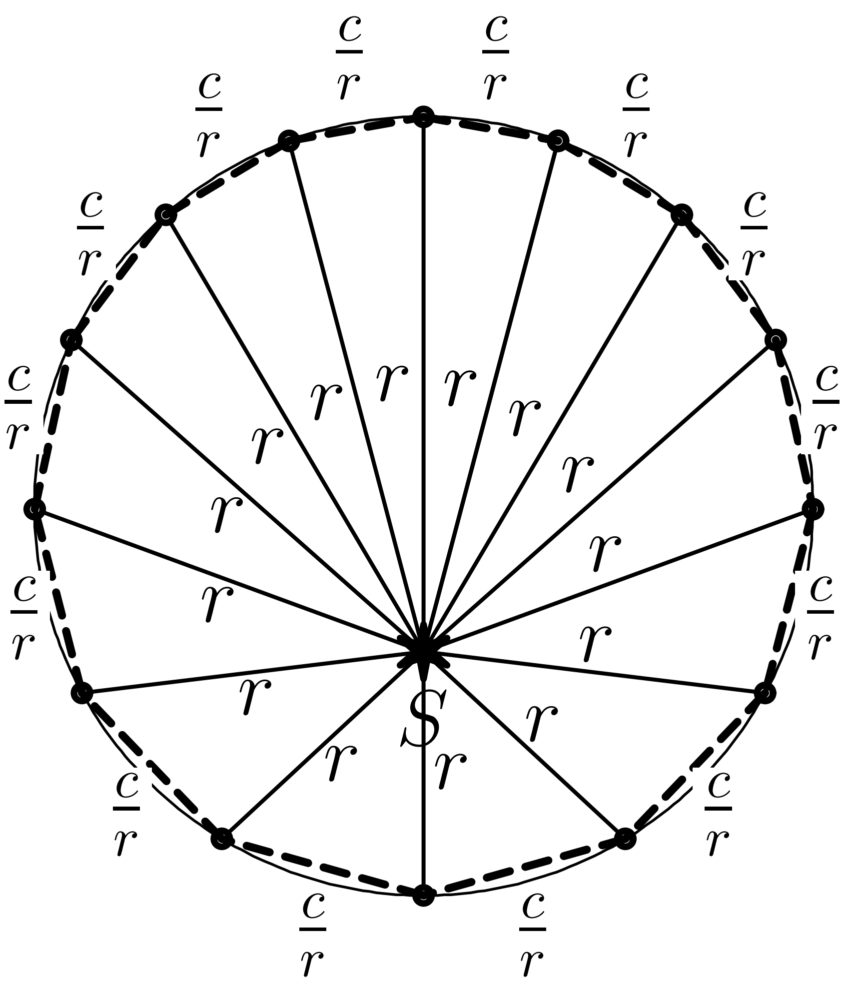
Kepler, Hypothesis vicaria:



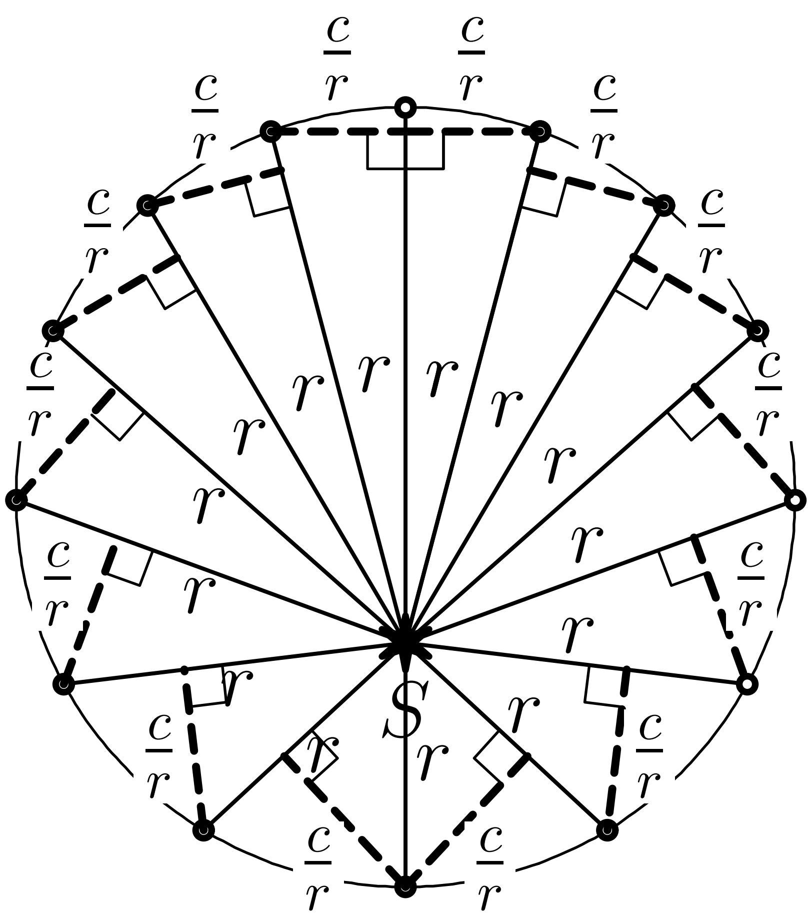

Kepler, Chap. 32-39:



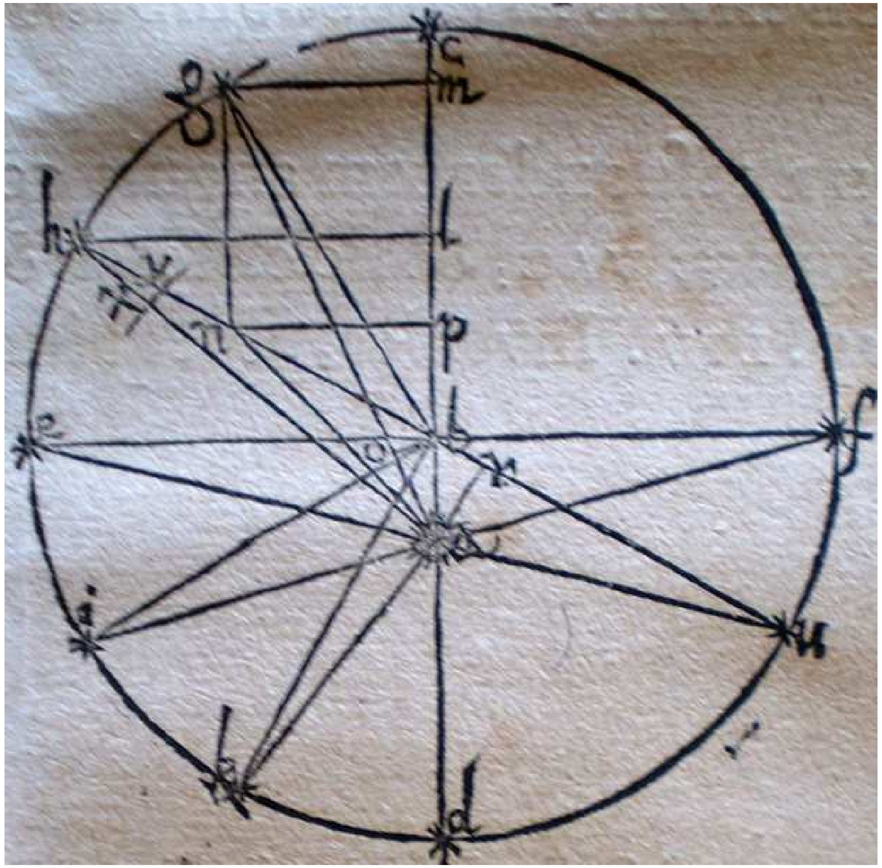
Kepler, Chap. 32-39:

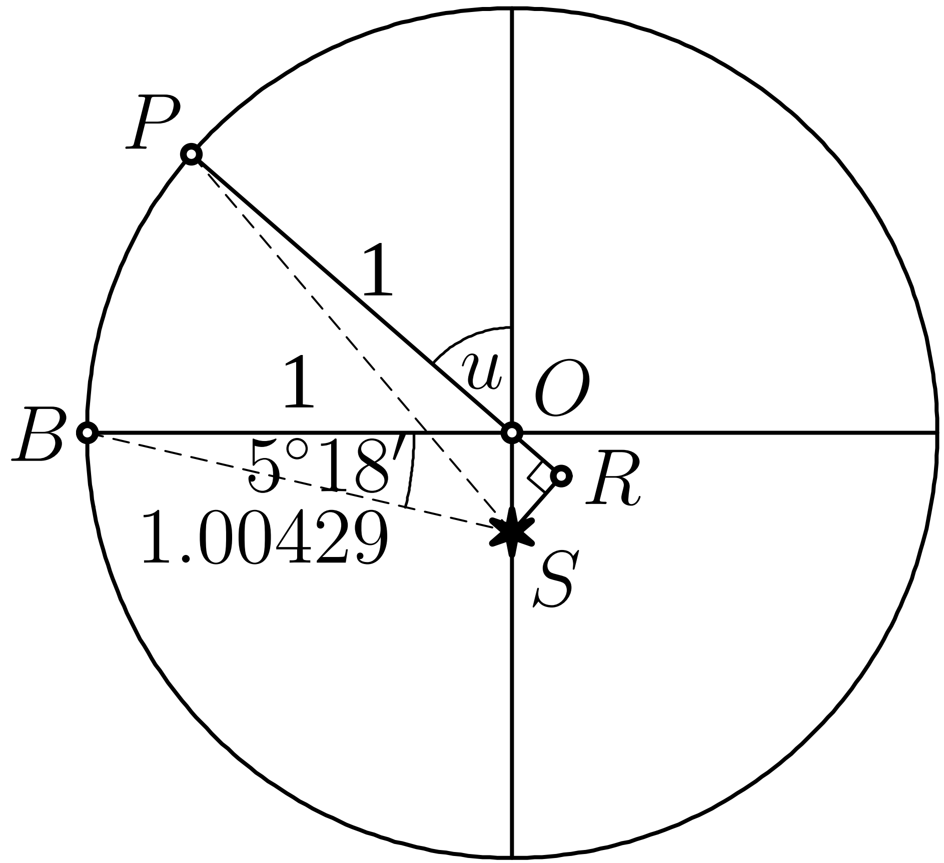



Kepler, Meth. imperf., Chap. 40:



# **Kepler's Pars IV :** The Great Idea in Chap. 56:

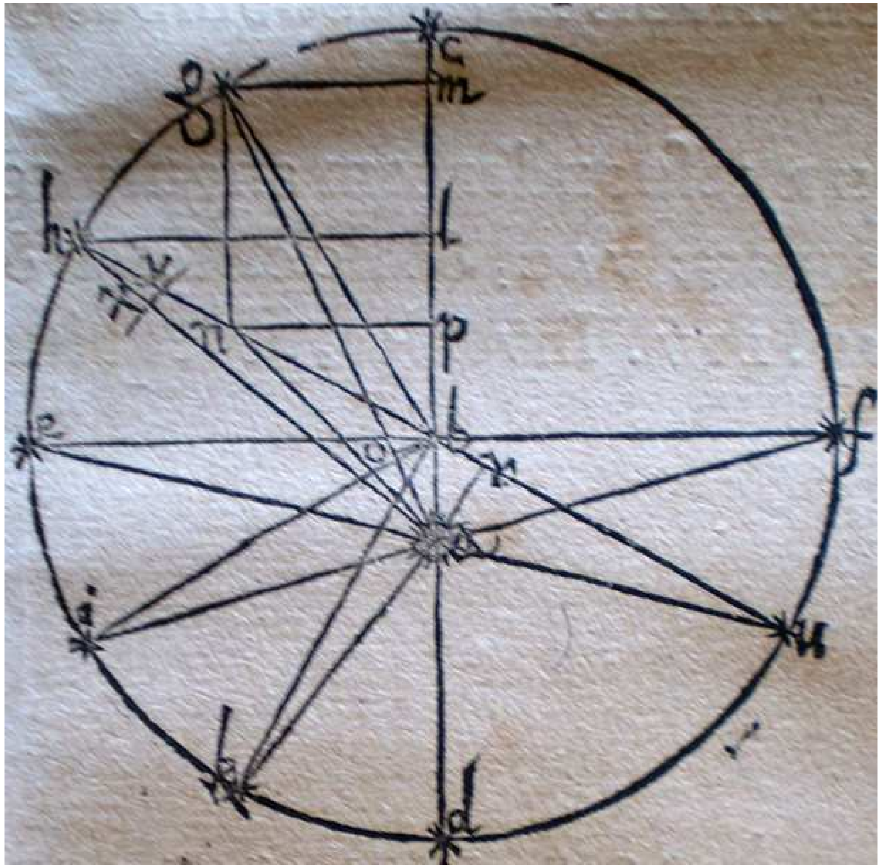


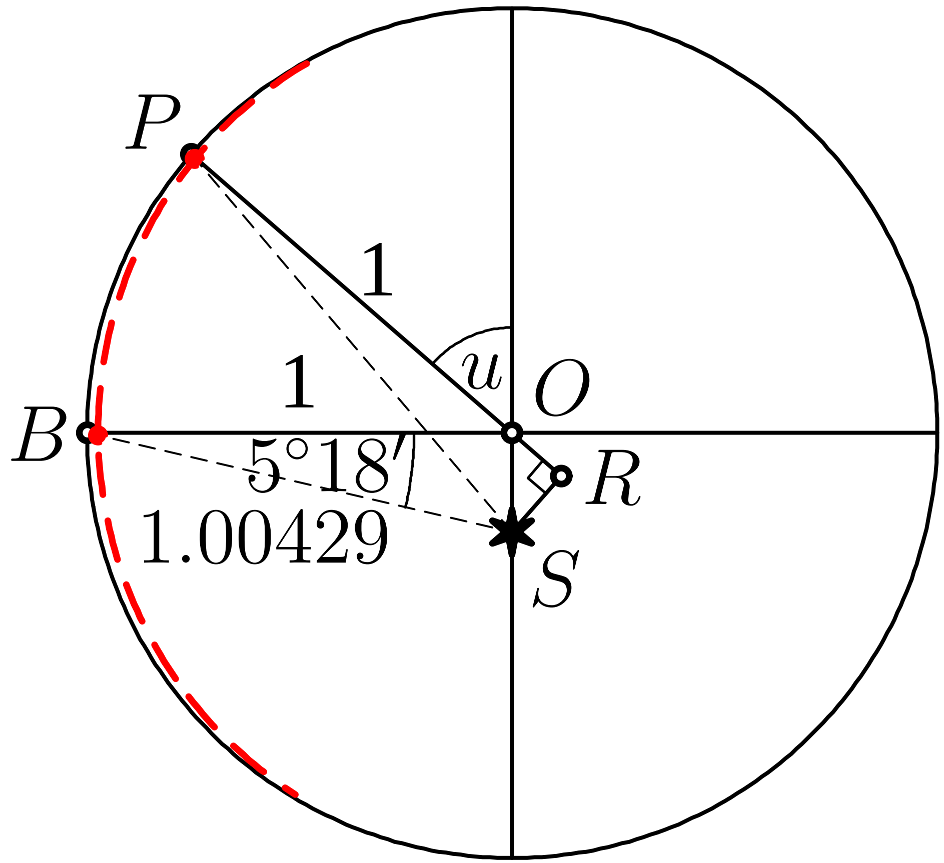



Obs.: Dist. $BS$ of Tycho's circle by factor $1.00429$ too large;
This value is (by chance) $1/\cos 5°18'$;

plane nihil dictum eſſe, itaque futilem fuiſſe meum de Marte trium-
phum; forte fortuito incido in ſecantem anguli 5. 18. quæ eſt menſura
æquationis Opticæ maximæ. Quem cum viderem eſſe 100429, hic
quaſi e ſomno expergefactus, & novam lucem intuitus, ſic cœpi ratio-

# **Kepler's Pars IV :** The Great Idea in Chap. 56:



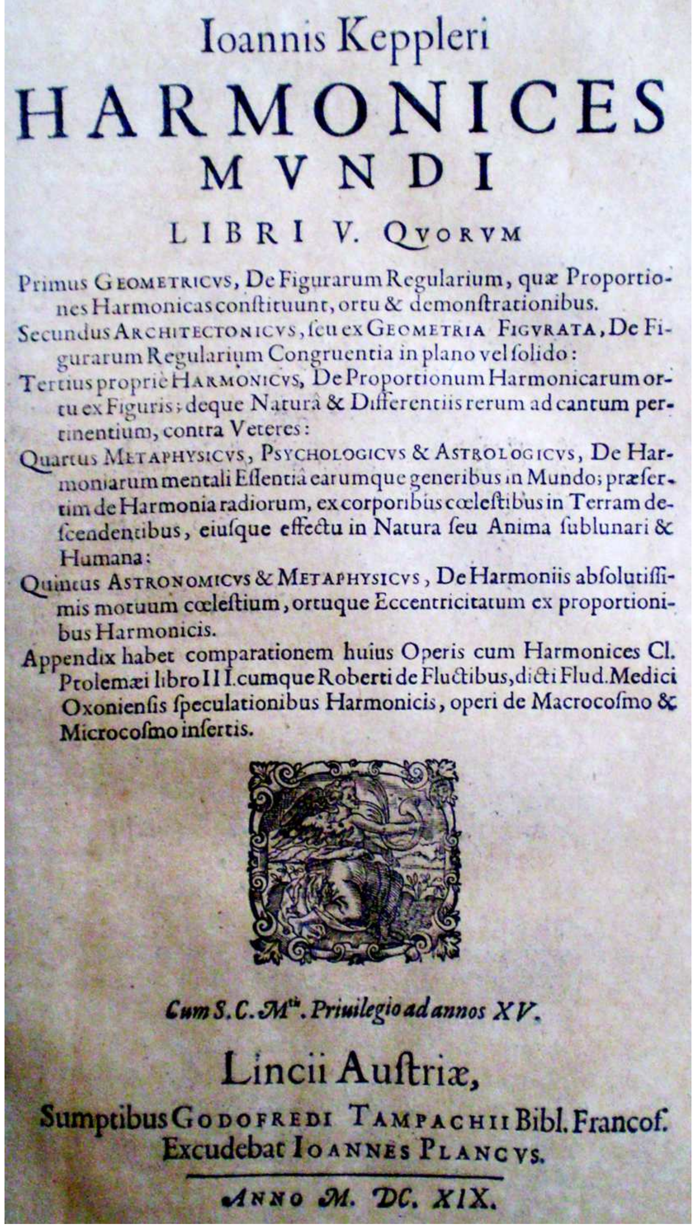




Idea: Replace 'hypoth.' by 'legs', $BS = BO$, $PS = PR$,...
and "awake from sleep"!!!

$PS = PR = 1 + e \cos u$ (Ch. 57–59) $\Rightarrow$ **orbit is ellipse !!**

Hoc jam obtento, non rationibus a priori, sed observationibus,... (This is now established, not from reasoning as before, but from observations...)

# **Kepler's Third Law** (1619):



Ioannis Keppleri

HARMONICES MVNDI

LIBRI V. QVORVM

Primus GEOMETRICVS, De Figurarum Regularium, quæ Proportiones Harmonicas constituunt, ortu & demonstrationibus.

Secundus ARCHITECTONICVS, seu ex GEOMETRIA FIGVRATA, De Figurarum Regularium Congruentia in plano vel solido:

Tertius propriè HARMONICVS, De Proportionum Harmonicarum ortu ex Figuris; deque Naturâ & Differentiis rerum ad cantum pertinentium, contra Veteres:

Quartus METAPHYSICVS, PSYCHOLOGICVS & ASTROLOGICVS, De Harmoniarum mentali Essentiâ earumque generibus in Mundo; præsertim de Harmonia radiorum, ex corporibus cœlestibus in Terram descendentibus, eiusque effectu in Natura seu Anima sublunari & Humana:

Quintus ASTRONOMICVS & METAPHYSICVS, De Harmoniis absolutissimis motuum cœlestium, ortuque Eccentricitatum ex proportionibus Harmonicis.

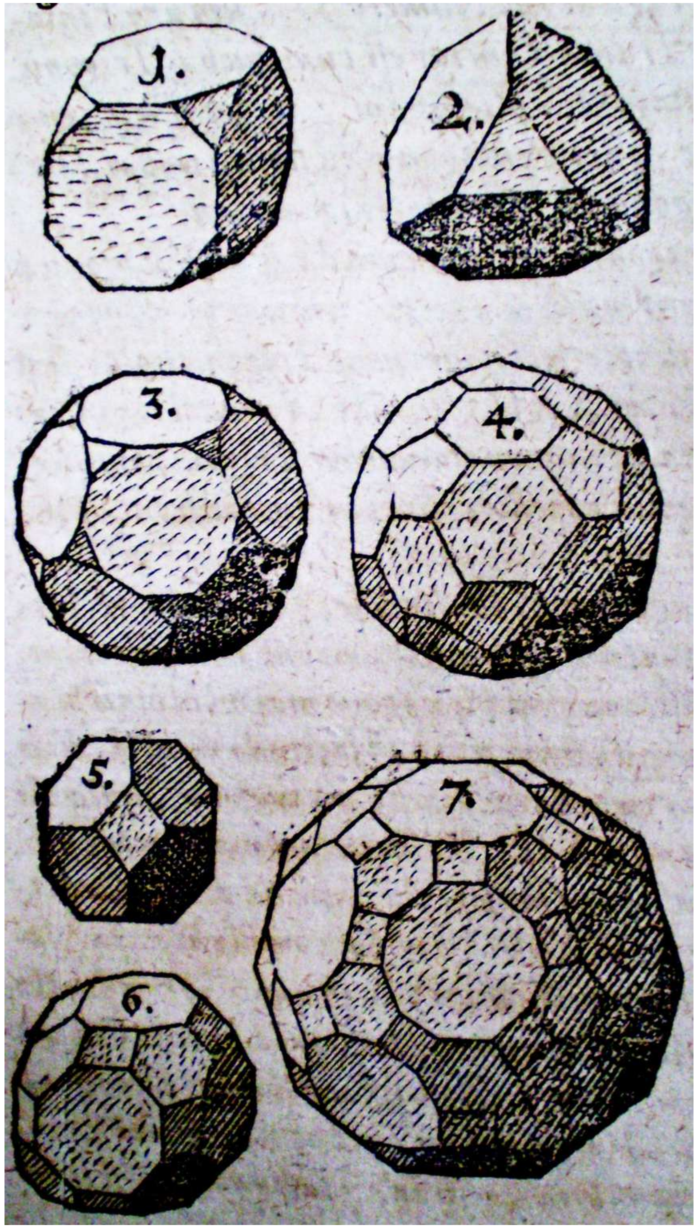
Appendix habet comparationem huius Operis cum Harmonices Cl. Ptolemæi libro III. cumque Roberti de Fluctibus, dicti Flud. Medici Oxoniensis speculationibus Harmonicis, operi de Macrocosmo & Microcosmo insertis.

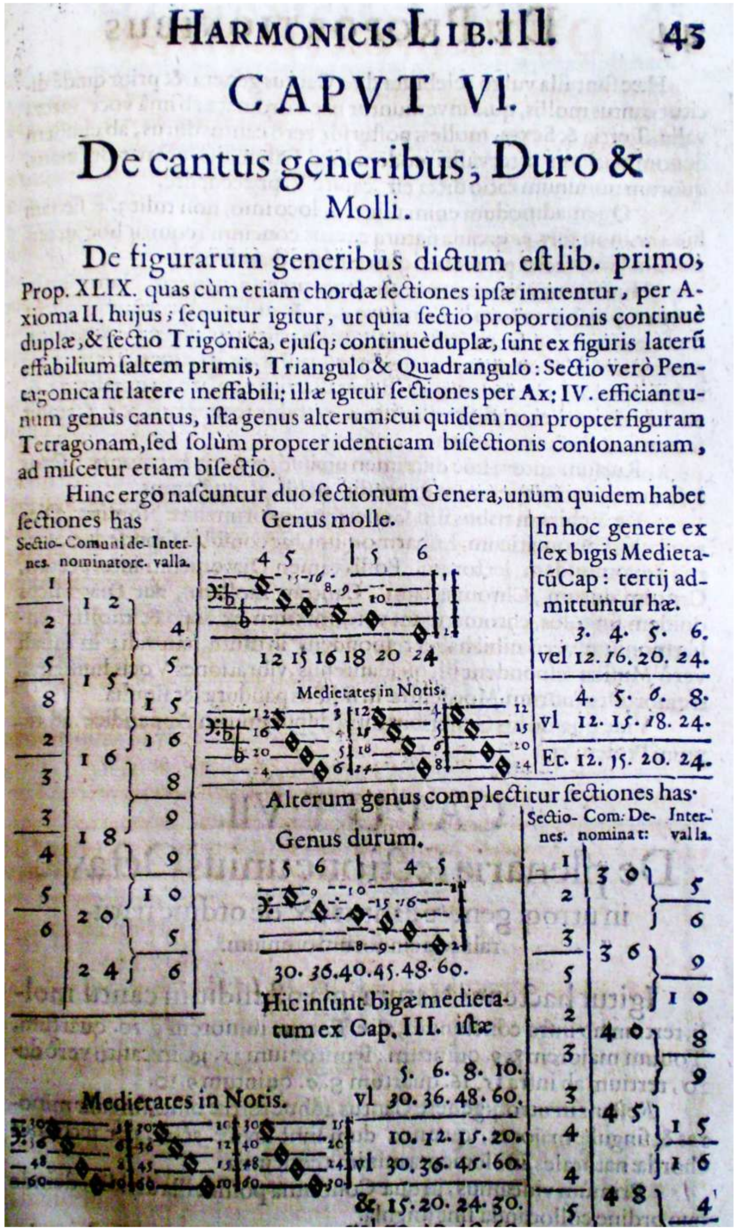
*Cum S. C. M^tis. Privilegio ad annos XV.*

Lincii Austriæ,

Sumptibus GODOFREDI TAMPACHII Bibl. Francof.
Excudebat IOANNES PLANCVS.

ANNO M. DC. XIX.



HARMONICIS LIB. III. 43

CAPVT VI.

De cantus generibus, Duro & Molli.

De figurarum generibus dictum est lib. primo, Prop. XLIX. quas cùm etiam chordæ sectiones ipsæ imitentur, per Axioma II. hujus; sequitur igitur, ut quia sectio proportionis continuè duplæ, & sectio Trigonica, ejusq; continuè duplæ, sunt ex figuris laterũ effabilium saltem primis, Triangulo & Quadrangulo: Sectio verò Pentagonica fit latere ineffabili; illæ igitur sectiones per Ax: IV. efficiant unum genus cantus, ista genus alterum: cui quidem non propter figuram Tetragonam, sed solùm propter identicam bisectionis consonantiam, ad miscetur etiam bisectio.

Hinc ergò nascuntur duo sectionum Genera, unum quidem habet sectiones has Genus molle.

In hoc genere ex sex bigis Medietatũ Cap: tertij admittuntur hæ.

Medietates in Notis.

Alterum genus complectitur sectiones has: Genus durum.

Hic insunt bigæ medietatum ex Cap. III. istæ

Medietates in Notis.

Search harmonies... from Geometry... and Music...

## ... Finally the result comes from vulgar numer. calculations ...

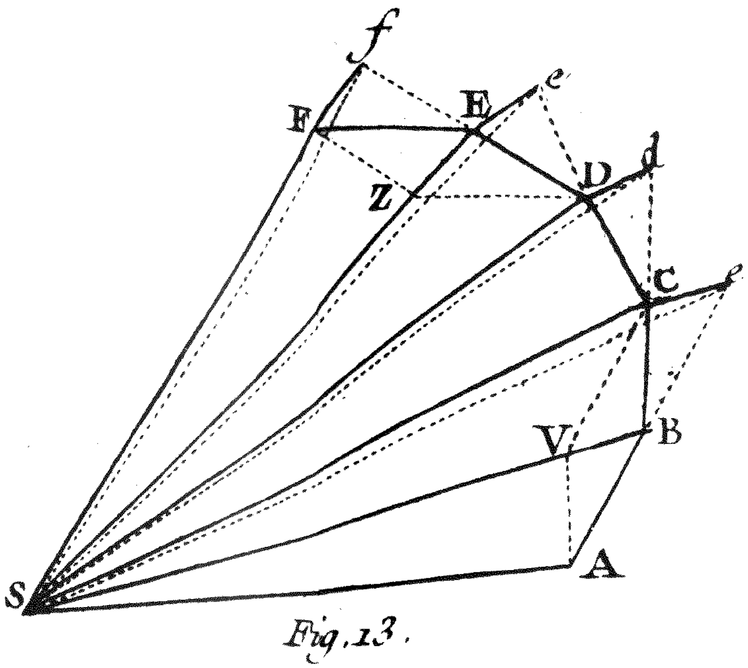
| | Diurni *Prim. Sec.* | Intervalla mediocria | Itinera diurna. |
|---|---|---|---|
| *Saturni Aphelij* | 1. 53. | 9510. | 1065 |
| *Perihelij* | 2. 7. | | 1208 |
| *Iovis Aphelij* | 4. 44 | 5200. | 1477 |
| *Perihelij* | 5. 15. | | 1638 |
| *Martis Aphelij* | 28. 44. | 1524. | 2627 |
| *Perihelij* | 34. 34. | | 3161 |
| *Telluris Apheliæ* | 58. 6. | 1000. | 3486 |
| *Periheliæ* | 60. 13. | | 3613 |
| *Veneris Apheliæ* | 95. 29. | 724 | 4149 |
| *Periheliæ* | 96. 50. | | 4207 |
| *Mercurij Aphelij* | 201. 0. | 388. | 4680 |
| *Perihelij* | 307. 3. | | 7148 |

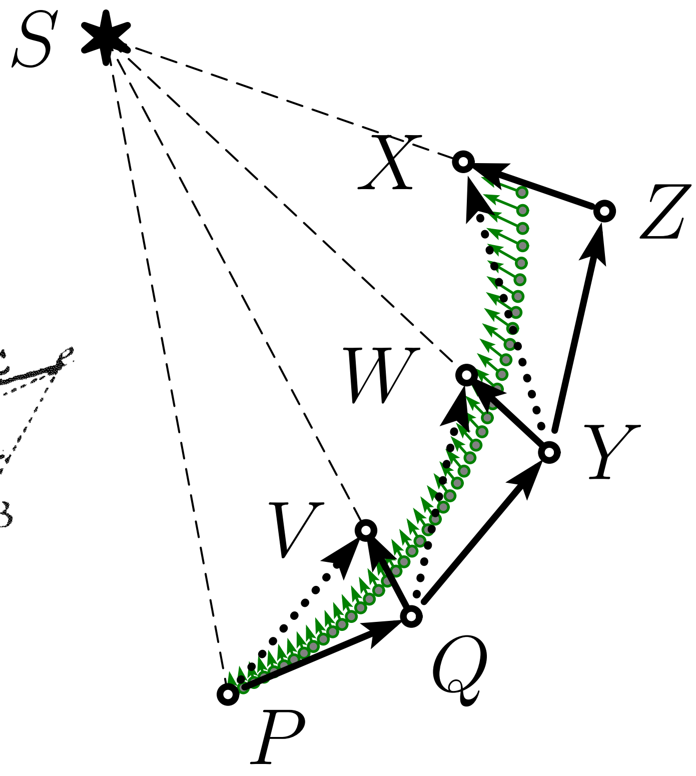
inter principia, primò crederem. Sed res est certissima exactissimaque, quòd *proportio quæ est inter binorum quorumcunque Planetarum tempora periodica, sit præcisè sesquialtera proportionis* mediarum distantiarum, id

"It is extremely certain and extremely exact that the ratio of the time period for two planets is one and a half of the ratio of the mean distances" (Lib. V, Caput 3, § 8).

**Isaac Newton.** (1684, 1687) Proof of Kepler 2:

... one of the most dramatic moments of the *real beginnings* was when Newton suddenly understood *so* much from *so* little... (R. Feynman, lecture of march 13, 1964.)



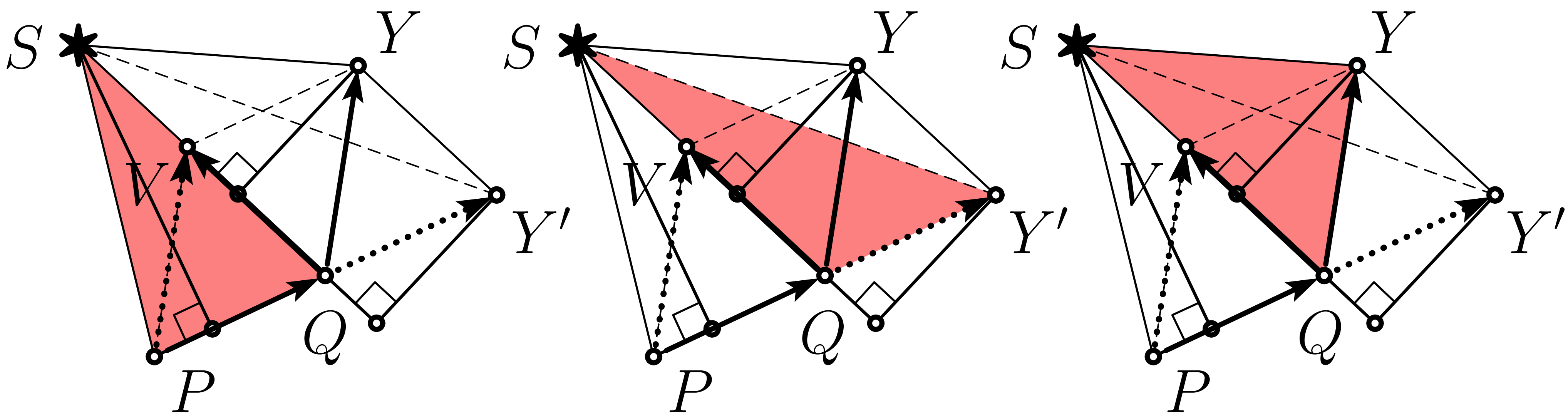

Idea:

contin. acting force

$\Rightarrow$ force impulses $f \cdot \Delta t$

at end of time steps

$=$ "sympl. Euler method".



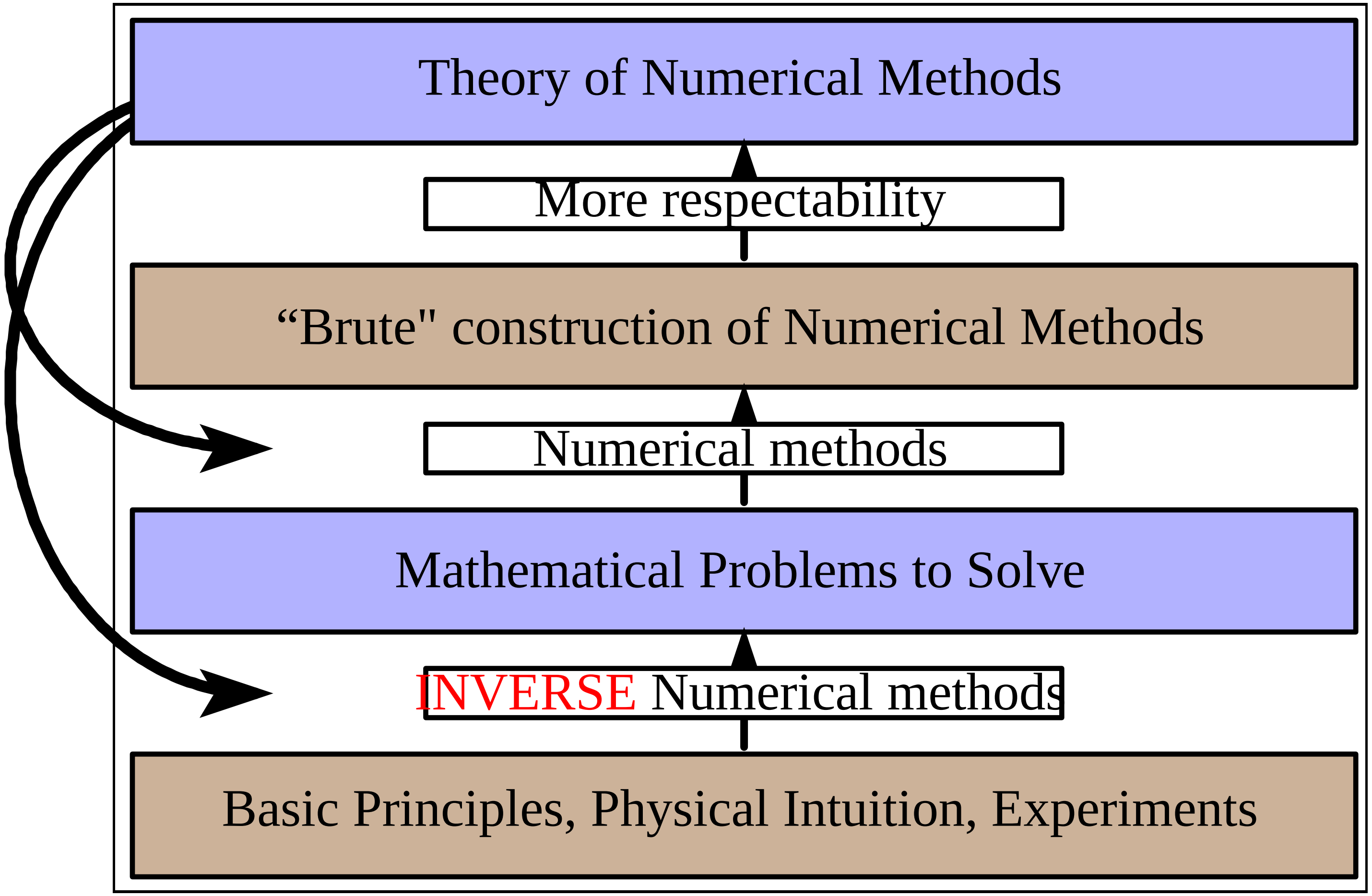

Eucl. I.41: All triangles have same area !

This became the "Theorema 1" of the *Principia*, and its proof is the first proof of modern science.

## Is this proof rigorous ???

The footnotes in Newton's *Mathematical Papers*, by D.T. Whiteside, trying to render this proof rigorous, are three times as long as Newton's original text ...

Not necessary !!

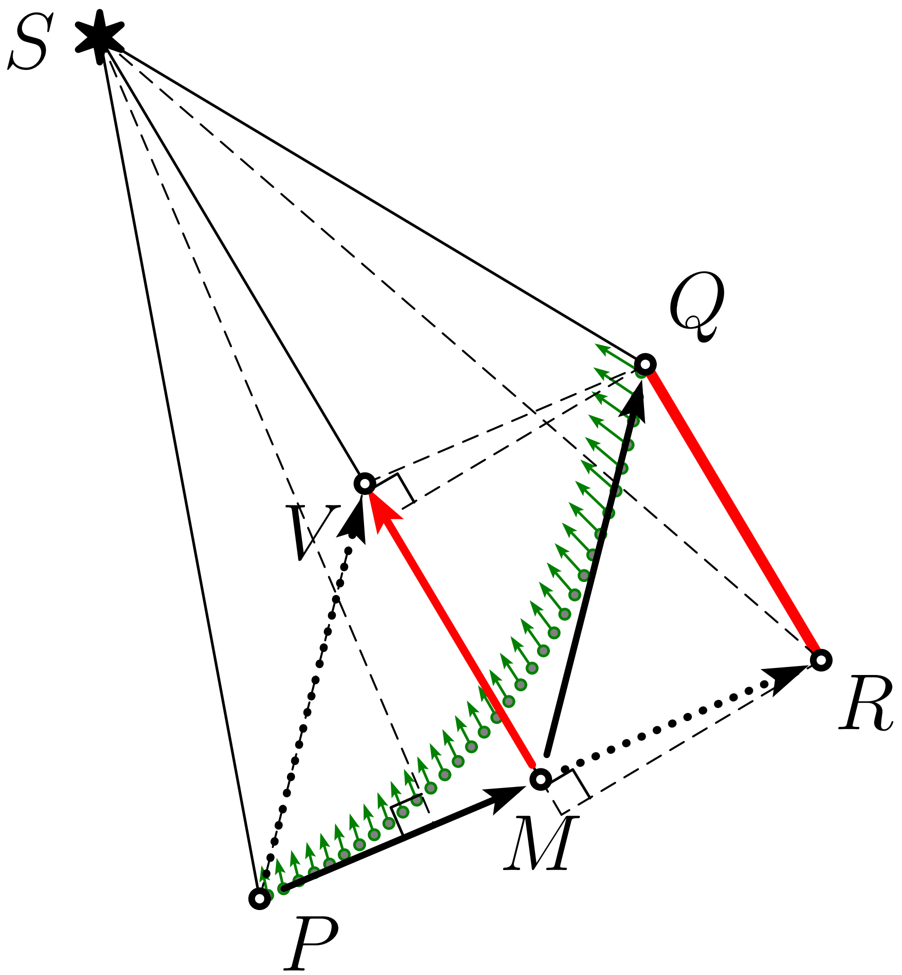



... because the theories up there apply to both situations.

We have a geometric numerical integrator
(exact preservation of angular momentum)

see: HLW, Geometric Numerical Integration 2002, 2006.

# Newton's Discovery of Gravitation Law from Kepler 1 & 2.




Idea:

place force impulse in midpoint $M$

("Störmer – Verlet")

$\Rightarrow$ force impulse $= RQ$

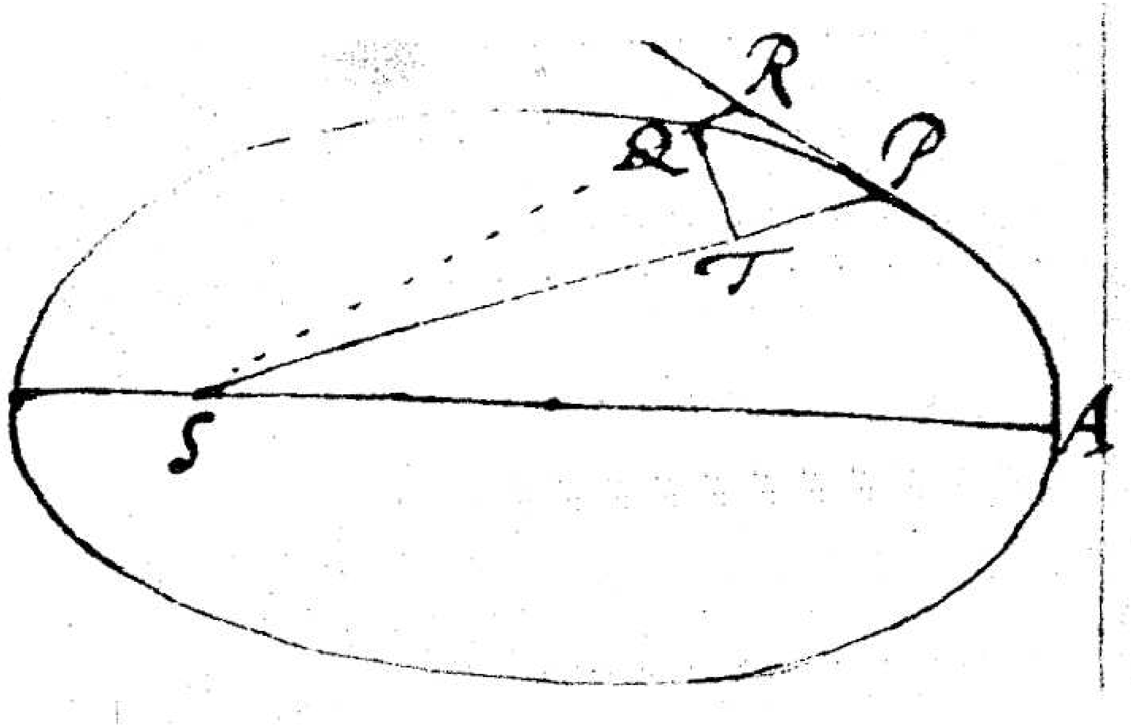
with $R$ on tangent and $Q$ on orbit.

## Newton's Lemma.

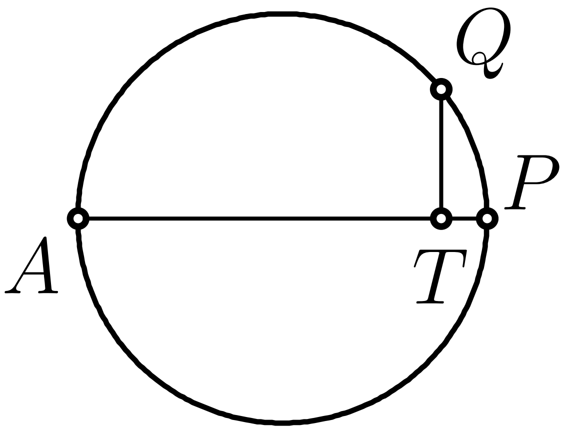
If orbit ellipse
and $S$ in focus,
then, for small step-sizes,

$RQ \approx \text{Const} \cdot QT^2$

where Const independent
of position of $P$.

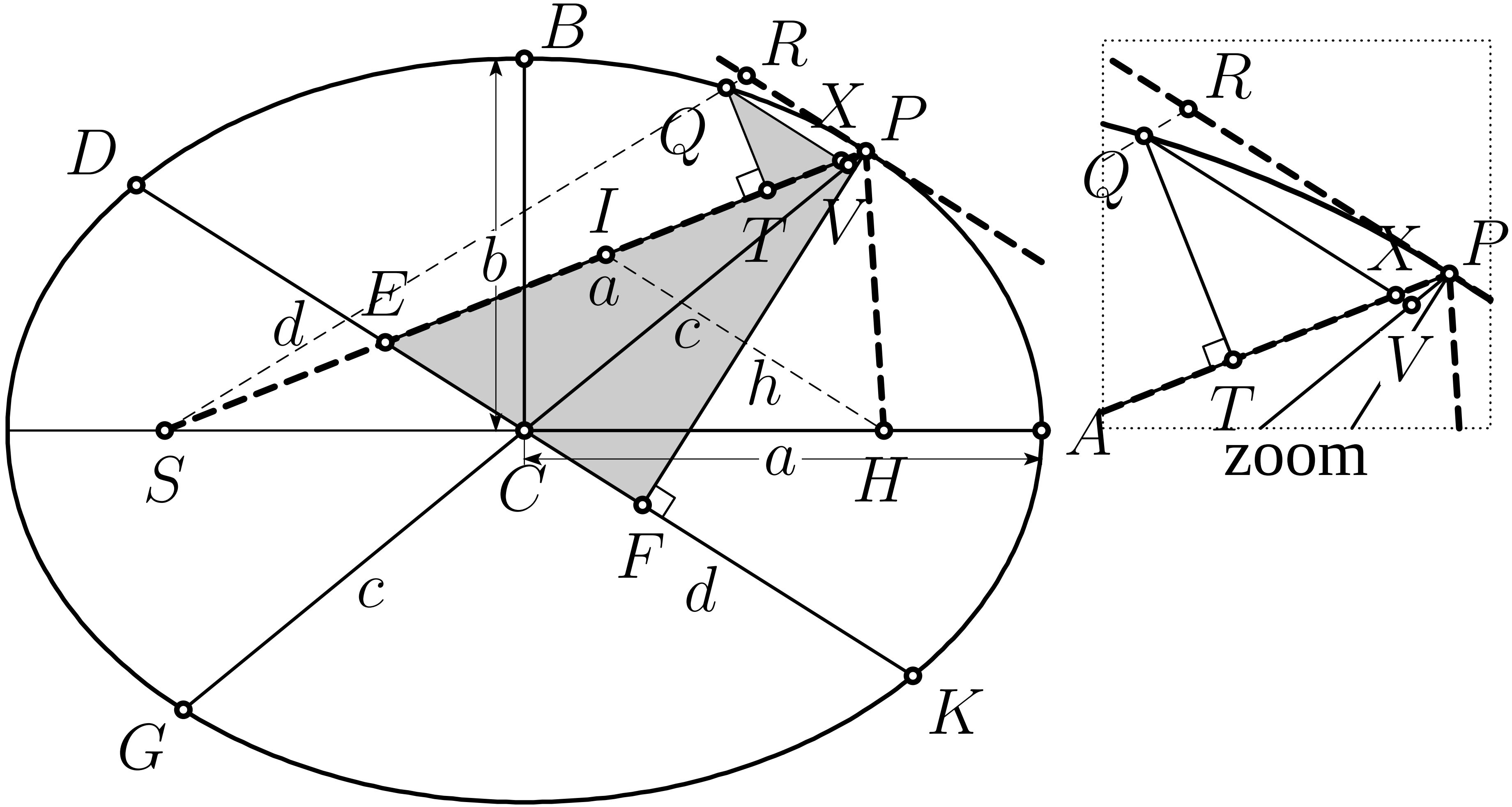



## Key to Proof:



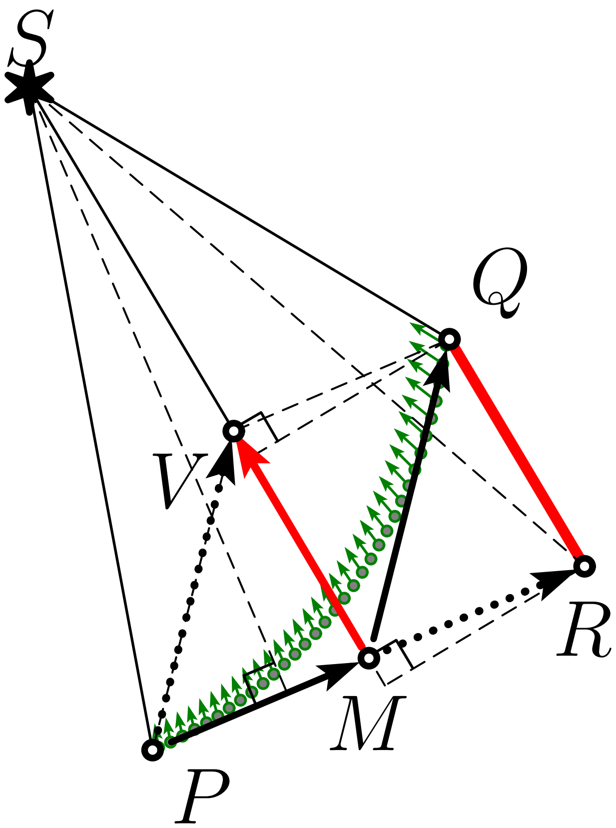

$AT \cdot TP = QT^2$ (Eucl. II.14)

In case of ellipse work with conjugate diameters:



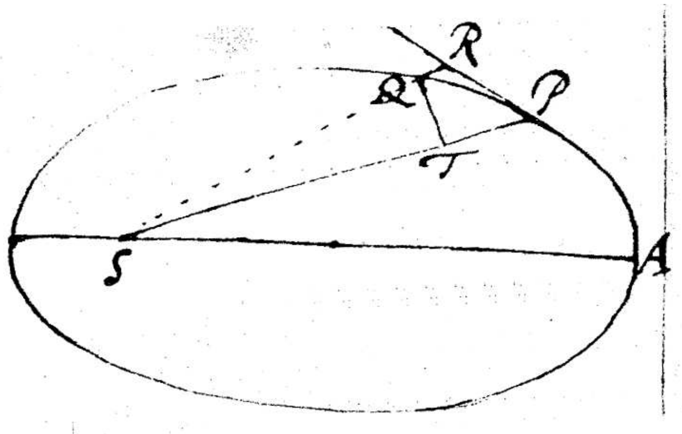

and use Apoll. II.6, Apoll. III.48, Eucl. I.6, Apoll. III.45, Apoll. III.52, Eucl. II.14, similar triangles, Thales and Apoll. VII.31.

# The Law of Gravitation.



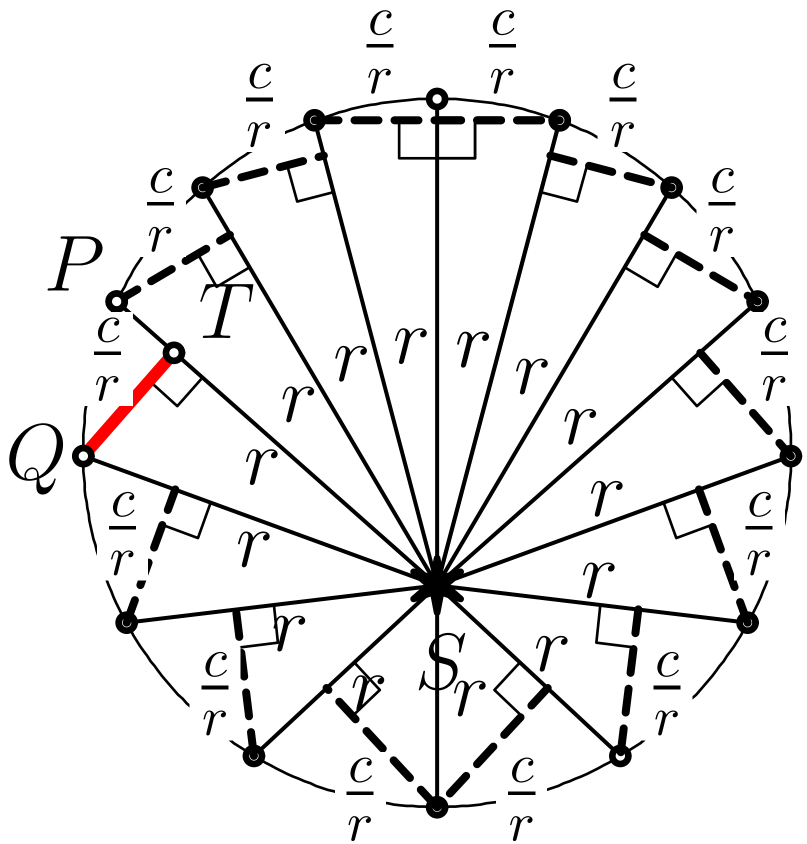

force prop. $RQ$

(Störmer-Verlet)



$RQ$ prop. $QT^2$

(Newton's Lemma)




$QT$ prop. $\frac{1}{r}$

(Kepler 2)

$$\Rightarrow \quad \text{force is proportional to } \frac{1}{r^2}$$

(Prop. XI of the Principia).

Another century later we are back to Euler (E122) 1747:

$$\text{I.}\ \frac{2ddx}{dt^2} = \frac{X}{M};\quad \text{II.}\ \frac{2ddy}{dt^2} = \frac{Y}{M};\quad \text{III.}\ \frac{2ddz}{dt^2} = \frac{Z}{M}$$

for which the above algorithms are inverse numerical methods !!

"While physicists call these "Newton's equations", they occur nowhere in the work of Newton or of anyone else prior to 1747."

"The discovery of this principle seems so easy, from Newtonian ideas, that is has never been attributed to anyone but Newton; such is the universal ignorance of the true history of mechanics."

(C. Truesdell, *Essays in the History of Mechanics*, 1968)

## **Une friandise:** Inverse square law $\Rightarrow$ Kepler 1 ??

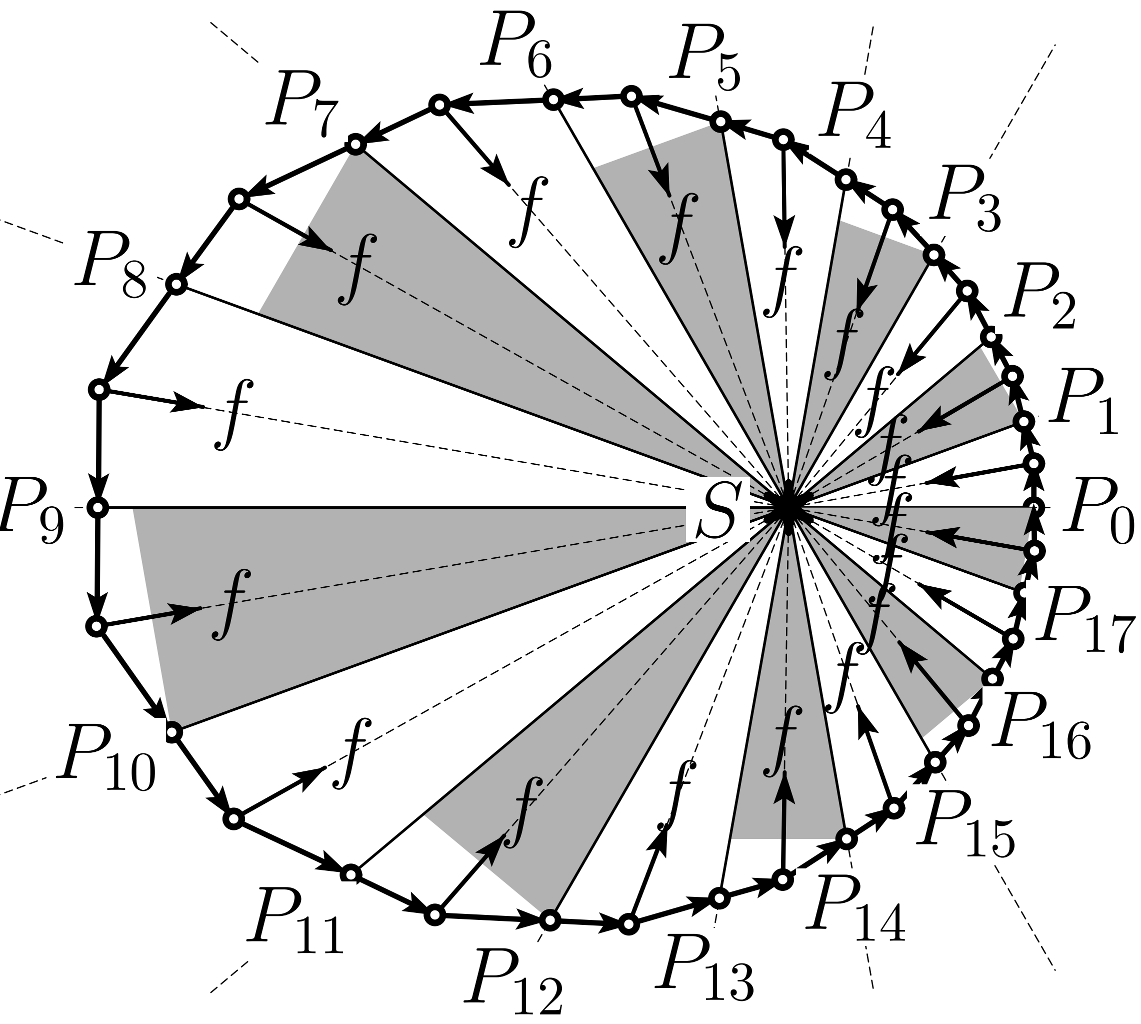
Pour voir présentement que cette courbe $ABC$ ... est toûjours une Section Conique, ainsi que Mr. Newton l'a supposé, *pag. 55. Coroll.I.* sans le démontrer; il y faut bien plus d'adresse:

(Joh. Bernoulli 1710)

... no calculus required, no differential equations, no conservation laws, no dynamics, no angular momentum, no constants of integration. This is Feynman at his best: reducing something seemingly big, complicated, and difficult to something small, simple, an easy.

(B. Beckman 2006)

**Feynman's idea** (1964, in a "lost lecture", rediscovered 1996):
*Equal Angles instead of Equal Time Steps.*




$\Delta\varphi$ constant;
(another step size strategy...)

$\Rightarrow$ areas of triangles prop. to $r^2$
(Eucl. VI.19);

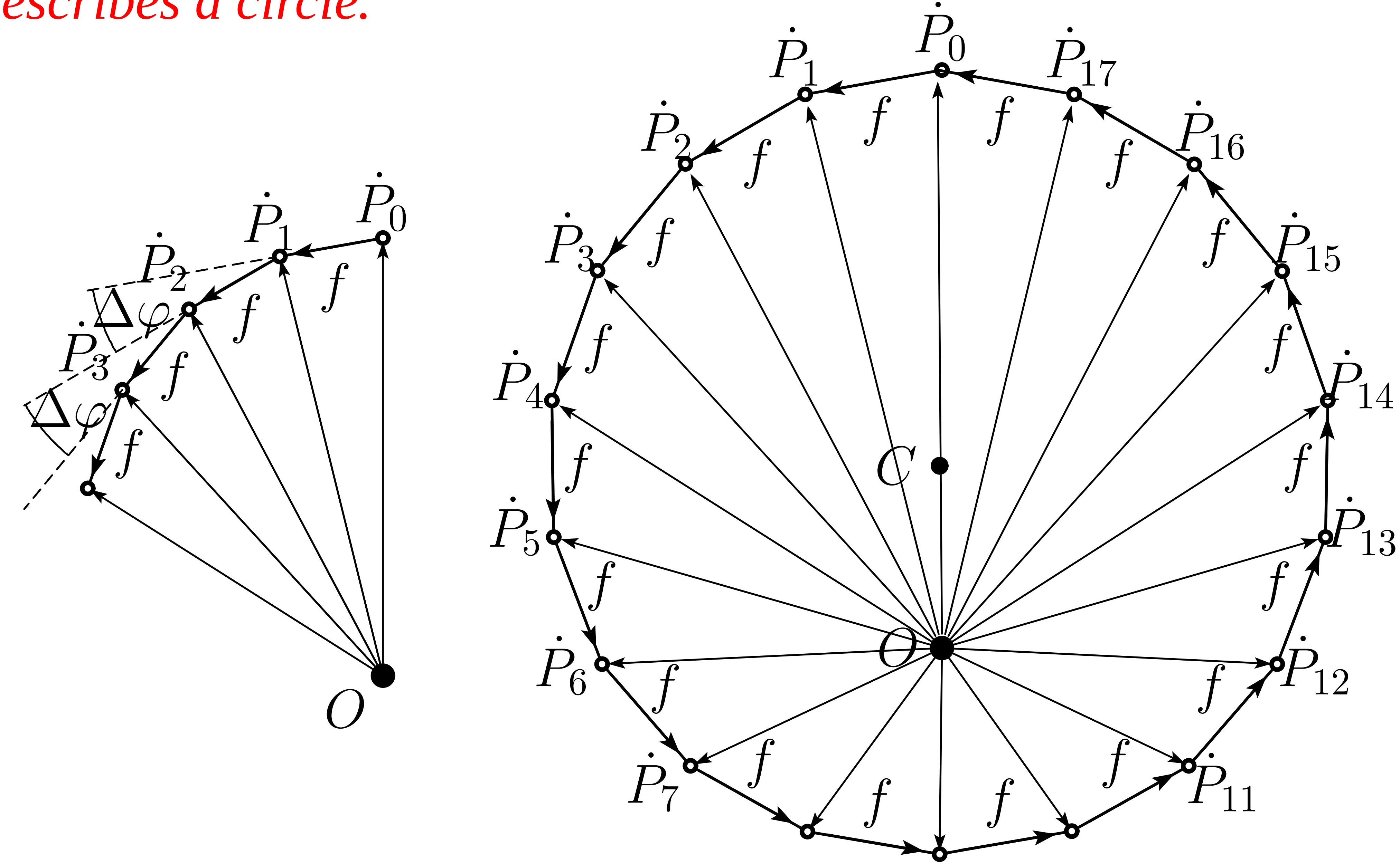
$\Rightarrow$ $\Delta t$ prop. to $r^2$ (Kepler 2);

centripetal force prop. to $\frac{1}{r^2}$
(hypothesis);

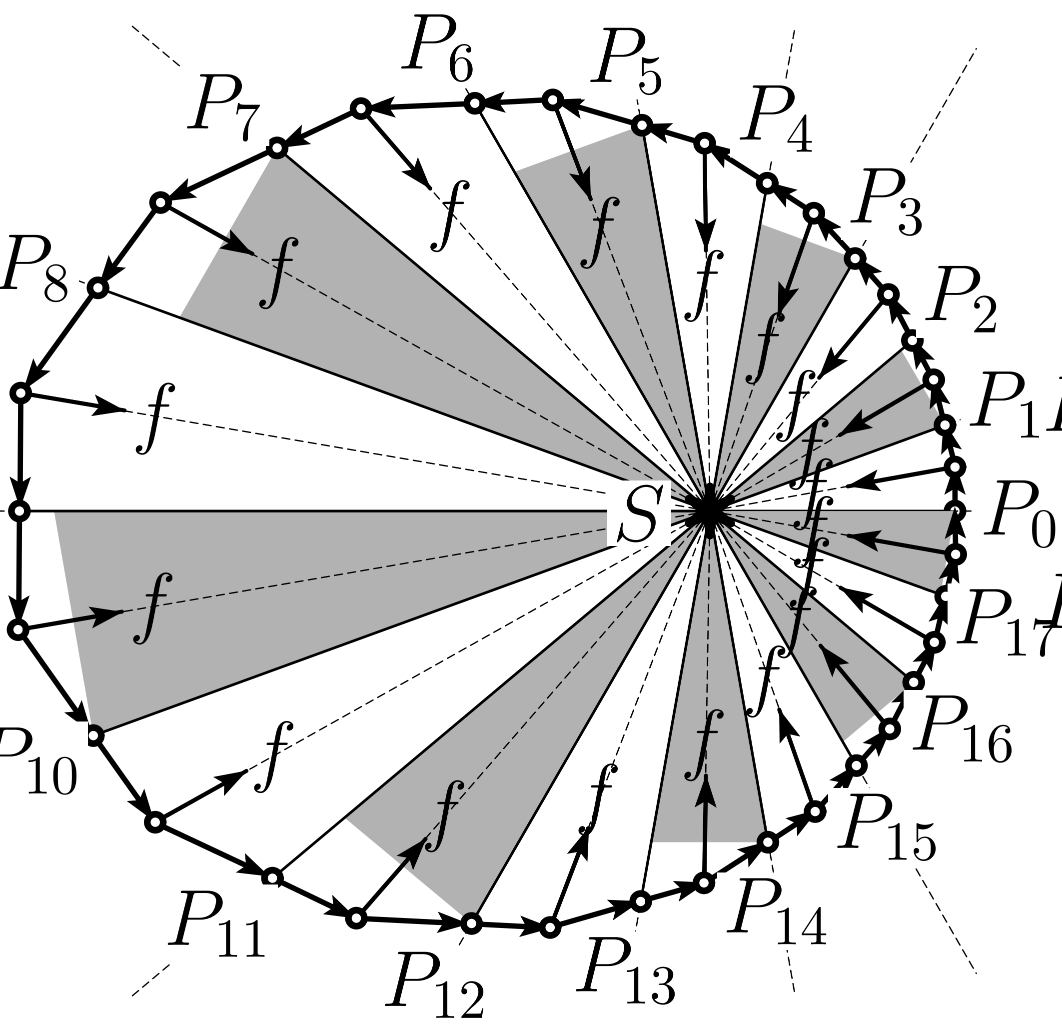
$\Rightarrow$ force impulses const. length;
directions of force impulses
vary regularly by constant $\Delta\varphi$;

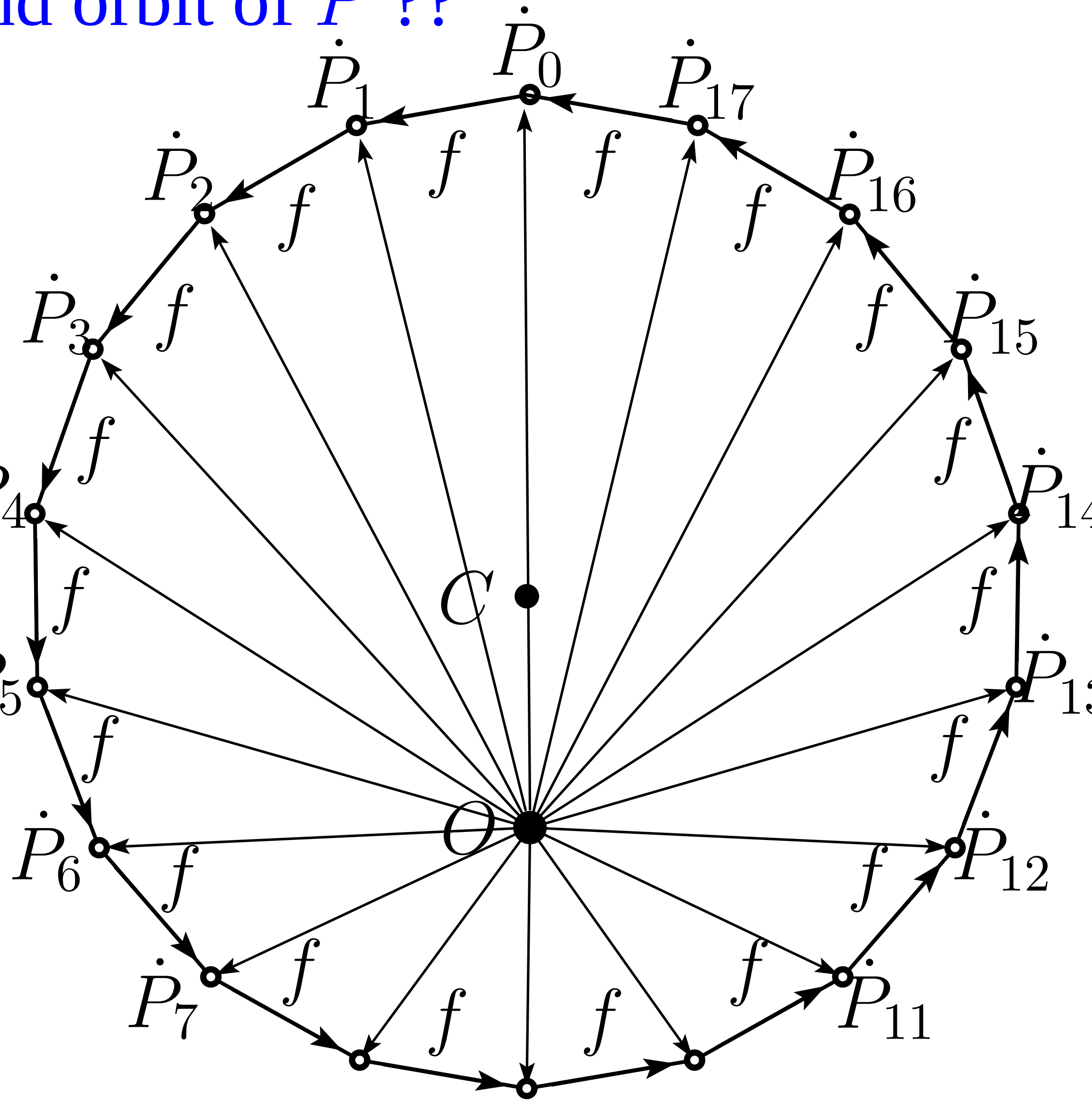
$\Rightarrow$ hodograph regular $n$-gon.

*Velocity $\dot{P}$ of planet orbiting under inverse square force describes a circle.*



Which relation between this and orbit of $P$ ??

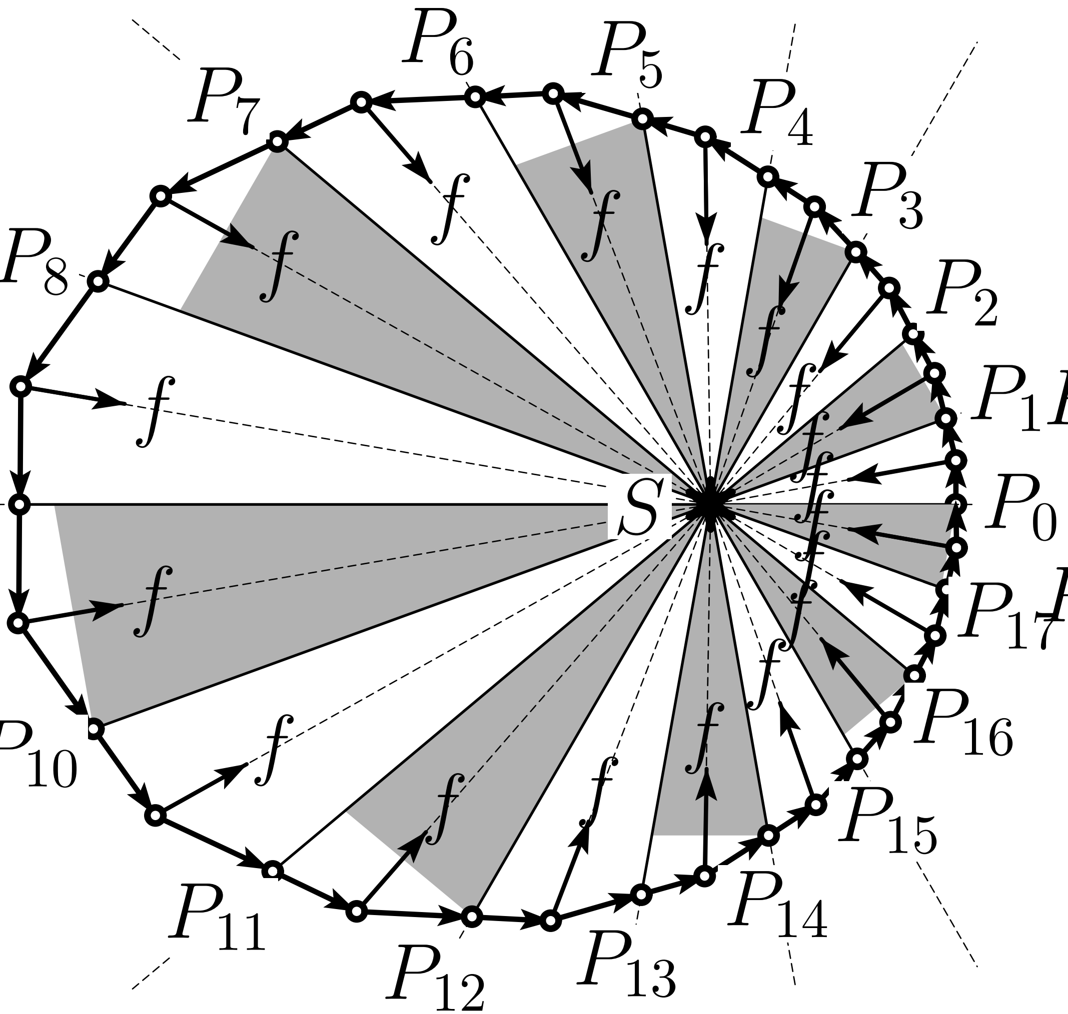


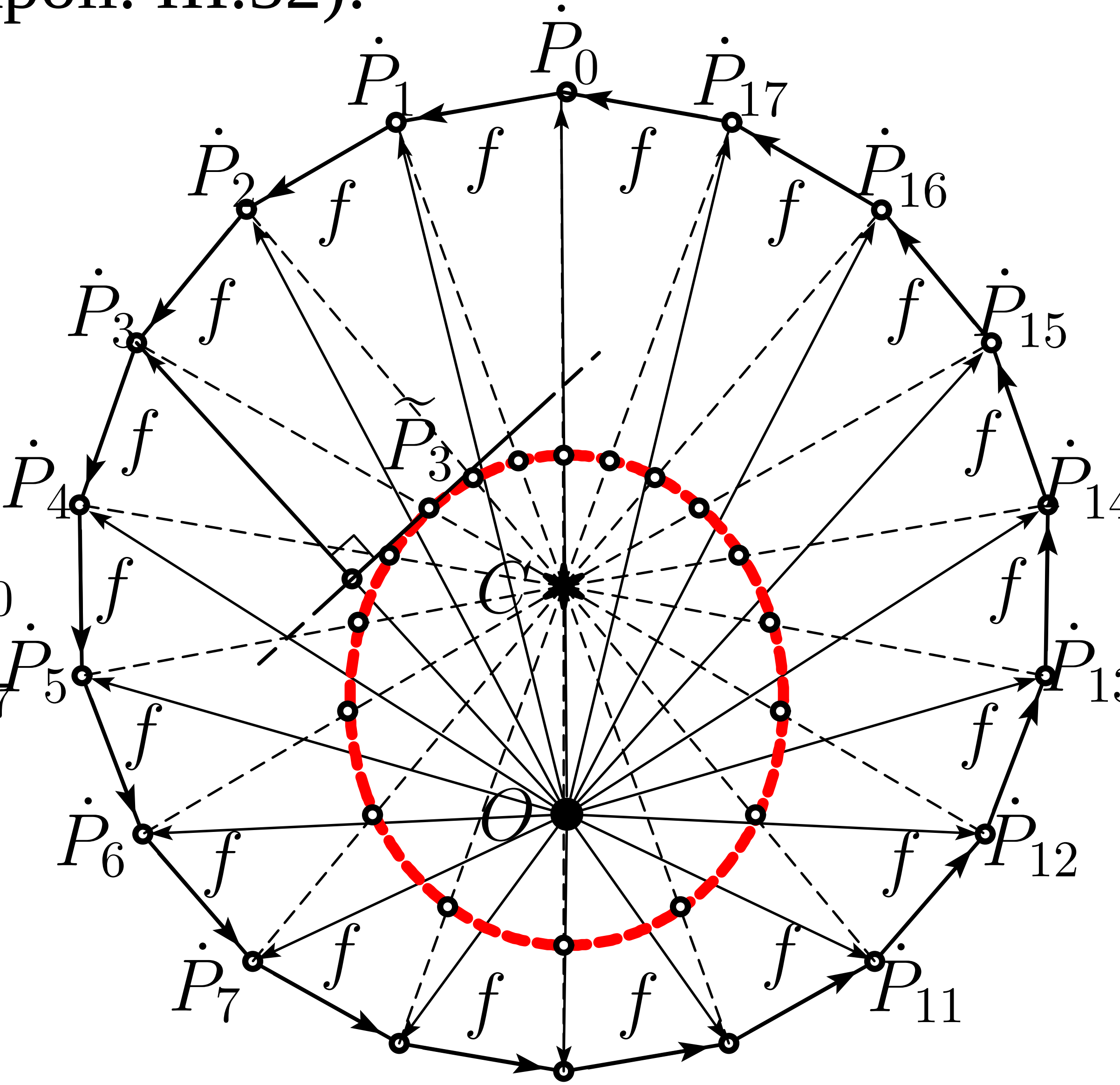




I took a long time to find that.

(R. Feynman 1964.)

**Idea:** *Consider curve of same distance from circle and origin $O$*
(which we know is ellipse; Apoll. III.52):




tangent in $P_i$ parallel to $O\dot P_i$; tangent in $\widetilde P_i$ orthogonal to $O\dot P_i$;
$\Rightarrow$ **orbit is ellipse !!** $\square$

# Épilogue venant du coeur ...

It is not easy to use the *geometric* method to discover things, it is very difficult, but the elegance of the demonstrations after the discoveries are made, is really very great. The power of the *analytic* method is that it is much easier to discover and to prove things, but not in any degree of elegance. There is a lot of dirty paper with x-es and y-s and crossed out cancellations and so on ... (laughers).

(R. Feynman, lecture of march 13, 1964, 35th minute.)

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**... ainsi s'achève mon enseignement à l'Uni de Genève ...**

**... il ne reste plus qu'à faire le BILAN ...**

# BILAN

- Président de la Section 1981-1984, 1997-2000;
- Président Soc. Math. Suisse 1998-1999;
- Président de la Classe Math. de l'Académie 2000-2006;
- Gründungsmitglied Plattform MAP der Akademie 2007;
- Swiss Delegate Internat. Math. Union 1998, 2002, 2006;
- The 2003 Peter Henrici Price, Sydney;
- Chair Scientific Progr. Comm.  ICIAM Zürich 2007;
- 70 lectures; 29 books, reeditions, transl.;
- 75 cities with invited talks;
- 40 oeuvres de musique chantées en concert;
- 518 sommets de montagne gravis.

... la

GRANDE DECEPTION DE MA VIE ...

**... la**
**GRANDE DECEPTION DE MA VIE ...**

**... n’avoir jamais dû refuser de devenir Doyen ou Vice-Doyen ...**

A D I E U

# A D I E U

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